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MÜNTZ—JACKSON TYPE THEOREMS VIA INTERPOLATION

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Let $\Lambda=\{\lambda_1, \lambda_2, \ldots\}$ be an infinite sequence of positive exponents. For $0\leqq a<1$ and $f(x)\in C[a, 1]$ put

$$E_{n,a}(f, \Lambda) = \min_{c_0, c_1, \ldots, c_n}{}^{*} \max_{a\leqq x\leqq 1} \left| f(x)-c_0-\sum_{k=1}^{n} c_k x^{\lambda_k}\right|$$

where $*$ indicates that $c_0=0$ has to be put if $a>0$. By the Müntz theorem, $\lim\limits_{n\to\infty} E_{n,0}(f, \Lambda)=0$ is true for each $f(x)\in C[0, 1]$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = +\infty. \tag{1}$$

CLARKSON and ERDŐS [1] generalized this theorem for any $0<a<1$. Suppose (1) is valid. The Müntz—Jackson type theorems (see e.g. D. NEWMAN [4]) are two-sided estimates concerning the rate of convergence of the sequences $E_{n,0}(f, \Lambda)$.

A possible way of obtaining an upper bound for $E_{n,a}(f, \Lambda)$ consists of two steps: first we choose an integer m and take the best approximating polynomial to $f(x)$ in $[a, 1]$ $P_m(x)=\sum\limits_{j=0}^{m} b_j x^j$, i.e. for which

$$\max_{a\leqq x\leqq 1} |f(x)-P_m(x)| = E_{m,a}(f) = \text{minimum}, \tag{2}$$

and then we approximate $P_m(x)$ term by term by linear combinations of the powers x^{λ_k} $(1\leqq k\leqq n)$.

For performing this second step we introduce a new, interpolation theoretical method which gives error estimates directly in the supremum norm.

LEMMA 1. *Let $\lambda\geqq 0$, $m\geqq 0$, and $0<x\leqq 1$ be real numbers, then*

$$\varepsilon = \left| x^{\lambda} - \sum_{k=1}^{n} x^{\lambda_k} \frac{2(\lambda+m)}{\lambda_k+\lambda+2m} \prod_{j=1}^{n} \frac{\lambda_k+\lambda_j+2m}{\lambda+\lambda_j+2m} \cdot \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{\lambda-\lambda_j}{\lambda_k-\lambda_j}\right| \leqq \frac{1}{x^m} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m}. \tag{3}$$

PROOF. For x fixed consider the function

$$\varphi_x(z) = \frac{x^z}{z+\lambda+2m} \prod_{j=1}^{n} (z+\lambda_j+2m)$$

of the complex variable z. $\varphi_x(z)$ is analytic in the half plane $\{\operatorname{Re} z\geqq -m\}$. Let

$p_x(z)$ denote the Lagrange-polynomial which interpolates $\varphi_x(z)$ at the nodes $\lambda_1, \lambda_2, \ldots, \lambda_n$. Explicity,

$$p_x(z) = \sum_{k=1}^{n} \frac{x^{\lambda_k}}{\lambda_k+\lambda+2m} \prod_{j=1}^{n} (\lambda_k+\lambda_j+2m) \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z-\lambda_j}{\lambda_k-\lambda_j}.$$

By a known formula (see e.g. WALSH [5]) the error of the interpolation is

$$\varphi_x(z)-p_x(z) = \frac{1}{2\pi i} \oint_C \frac{\varphi_x(z)}{\zeta-z} \prod_{k=1}^{n} \frac{z-\lambda_k}{\zeta-\lambda_k}\, d\zeta, \tag{4}$$

where C is an arbitrary rectifiable closed Jordan-curve lying in $\{\mathrm{Re}\, z \geqq -m\}$ to which $z, \lambda_1, \ldots, \lambda_n$ are interior. We put $C=C_R$, where

$$C_R = \{\zeta\colon \mathrm{Re}\,\zeta = -m,\ |\mathrm{Im}\,\zeta| \leqq R\} \cup \{\zeta\colon \mathrm{Re}\,\zeta > -m,\ |\zeta+m| = R\} = C_R' \cup C_R''$$

and let $R\to\infty$. On the half-circle C_R'' we have $\varphi_x(\zeta)=O(R^{n-1})$ while the kernel of integration is $O(R^{-n-1})$ $(R\to\infty)$. Hence it follows that $\lim\limits_{R\to\infty} \int_{C_R''} = 0$, and by (4)

$$\varphi_x(z)-p_x(z) = \frac{1}{2\pi i} \int_{-m-i\infty}^{-m+i\infty} \frac{\varphi_x(\zeta)}{\zeta-z} \prod_{k=1}^{n} \frac{z-\lambda_k}{\zeta-\lambda_k}\, d\zeta \tag{5}$$

holds for any z in $\{\mathrm{Re}\, z > -m\}$. It follows from the definitions of $\varphi_x(z)$ and $p_x(z)$ that

$$\varepsilon = \frac{2(\lambda+m)}{\prod_{j=1}^{n}(\lambda+\lambda_j+2m)} |\varphi_x(\lambda)-p_x(\lambda)|,$$

therefore (5) implies (with $z=\lambda$)

$$\varepsilon = \frac{2(\lambda+m)}{\prod_{j=1}^{n}(\lambda+\lambda_j+2m)} \left| \frac{1}{2\pi} \int_{-m-i\infty}^{-m+i\infty} \frac{\varphi_x(\zeta)}{\zeta-\lambda} \prod_{k=1}^{n} \frac{\lambda-\lambda_k}{\zeta-\lambda_k}\, d\zeta \right| \leqq \tag{6}$$

$$\leqq \frac{\lambda+m}{\pi} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m} \int_{-m-i\infty}^{-m+i\infty} \left| \frac{x^\zeta}{(\zeta-\lambda)(\zeta+\lambda+2m)} \prod_{k=1}^{n} \frac{\zeta+\lambda_k+2m}{\zeta-\lambda_k} \right| |d\zeta|.$$

Now we notice that for $\mathrm{Re}\,\zeta = -m$ we have

$$|x^\zeta| = x^{-m}, \quad |\zeta-\lambda| = |\zeta+\lambda+2m|, \quad |\zeta+\lambda_k+2m| = |\zeta-\lambda_k| \qquad (1 \leqq k \leqq n)$$

further

$$\int_{-m-i\infty}^{-m+i\infty} \frac{|d\zeta|}{|\zeta-\lambda|^2} = \int_{-\infty}^{\infty} \frac{dy}{y^2+(\lambda+m)^2} = \frac{\pi}{\lambda+m}. \tag{7}$$

(6) and the facts above prove (3).

COROLLARY.

$$E_{n,a}(x^\lambda, \Lambda) \leqq \frac{1}{a^m} \prod_{k=1}^{n} \frac{|\lambda-\lambda_k|}{\lambda+\lambda_k+2m}. \tag{8}$$