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ON THE POTENCY OF SPACES WITH GENERALIZED DISPERSION POINTS UNDER MAPS AND FUNCTIONS

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In [1] C. A. COPPIN proved the following:

THEOREM (Coppin). *Let X be a connected, locally connected Hausdorff space and Y a connected Hausdorff space with a dispersion point. Then any map from X to Y is necessarily a constant.*

The following represents a slight improvement of Coppin's result.

THEOREM 1. *Let X be a connected, locally connected space and Y a connected T_1-space with a dispersion point. If $f: X\to Y$ is a connected function from X to Y such that f has closed point inverses, then f is a constant.*

REMARK. Coppin assumed both X and Y are Hausdorff and that f is continuous. His proof, however, shows that it is enough to assume that Y is T_1 and that f takes connected sets onto connected sets and f has closed point inverses. Connected functions have the property that $f(\overline{C})\subset\overline{f(C)}$ for each connected set C in X whenever Y is T_1 ([2], Theorem 3). On the other hand, requiring f to have closed point inverses entitles us to conclude $f^{-1}(Y-\{p\})$ is an open set in X because $f^{-1}(Y-\{p\})=X-f^{-1}(p)$ is an open set in X. The proof of the theorem given in [1] now goes through unchanged for Theorem 1.

As an example of a non-continuous connected function with closed point inverses, consider the following. Let X=the Sorgenfrey line, Y=the real line. Define $f: X\to Y$ by

$$f(x)=\begin{cases} x & \text{if } x\leqq 4 \\ -x+7 & \text{if } x>4. \end{cases}$$

We now prove the following companion theorem to the theorem of Coppin.

THEOREM 2. *Let X be a space with a generalized dispersion point p and assume also that X is dense-in-itself (each point of X is a limit point of X). Let Y be a locally connected T_1-space. If $f: X\to Y$ is a map such that f^{-1} preserves connected sets, then f is a constant. Let X be dense-in-itself and p a point in X. If $X-\{p\}$ is totally disconnected then p is a generalized dispersion point of X.*

PROOF. Suppose $f\neq$constant. Then there exists $a\in X-\{p\}$ such that $f(a)\neq f(p)$. Let $b=f(a)$ and $q=f(p)$. Pick an open set V in Y containing b such that $q\notin V$. Since Y is locally connected, we may assume that V is also connected. Note that $U=f^{-1}(V)$ is an open set containing a. Hence there exists $a'\in U$ such that $a'\neq a$. Also $U\subset X-\{p\}$, hence, being connected, U must be a singleton.

This contradiction (that U is a singleton containing two distinct points a and a') shows that f must be a constant.

REMARK. Let us re-emphasize that X is *not* assumed to be connected in Theorem 2 so that the theorem is applicable, for example, if X is the Sorgenfrey line with $p \in X$ arbitrary.

By Lemma 1 of [2], we know that a non-degenerate T_1 connected space is dense-in-itself. Hence we have the following:

COROLLARY 1. *Let X be a non-degenerate connected T_1-space with a dispersion point and let Y be a locally connected T_1-space. If $f: X \to Y$ is a map such that f^{-1} preserves connected sets, then f is a constant.*

REMARK. Theorem 2 of [5] states that if $f: X \to Y$ is a closed function such that $f^{-1}(y)$ is connected for each $y \in Y$, then f^{-1} preserves connected sets. Hence we have a second corollary to Theorem 2.

COROLLARY 2. *Let X be a non-degenerate connected T_1-space with dispersion point and let Y be a locally connected T_1-space. If $f: X \to Y$ is a closed map such that $f^{-1}(y)$ is connected for each $y \in Y$, then f is a constant.*

REMARK. Let X be the example of KNASTER and KURATOWSKI [3] of a connected space in the plane with a dispersion point. Then X is a completely regular Hausdorff space and so there exist non-constant continuous functions from X onto $[0, 1]$. Hence the hypothesis that f^{-1} preserves connected sets cannot be dropped. The proof of Theorem 2, however, shows that if Y is regular, it is enough to require that $f^{-1}(V)$ be a connected set in X for each connected *closed* set V in Y. Functions f which have the stronger property that f^{-1} preserves closed connected sets have been called semiconnected (see [5], for example).

Incidentally, Theorem 1 of [5] is a much weaker result than Theorem 9 of [2], so that it is wise to consult [2] before launching a program to investigate generalizations of or sufficient conditions for continuity.

Certainly the condition that f^{-1} preserves connected sets has some force even in the absence of continuity. This suggests a theorem without the continuity assumption.

THEOREM 3. *Let $f: X \to Y$ be a function from a space X with generalized dispersion point p to an arbitrary topological space Y. If f^{-1} preserves connected sets, then either f is constant or $f(X)$ has a generalized dispersions point.*

PROOF. Let $f(p) = q$. Then $B = f^{-1}(q)$ is connected and $f|X - B$ is a $1-1$ function onto $f(X) - \{f(p)\}$ since f^{-1} preserves connected sets, and $f(X) - \{f(p)\}$ is totally disconnected or void. Consequently either $f(X) = \{f(p)\}$ or $f(p)$ is a generalized dispersion point of $f(X)$.

Let E be the category of topological spaces whose morphisms are the functions of Theorem 3.

COROLLARY 3. *There exists a category $F \subset E$ that is a non-full subcategory with morphisms that are epimorphisms and whose objects are points or spaces with generalized dispersion points.*