ON FULLY DECOMPOSABLE ABELIAN TORSION GROUPS

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§ 1. Introduction

A group is said to be fully decomposable, if it can be represented as the direct sum of directly indecomposable groups, the latter being groups which do not decompose into the direct sum of two of its proper subgroups. As we know that among the abelian torsion groups only the groups $C(p^m)$ ($0 \leqq m \leqq \infty$) are directly indecomposable ([3], [6][1]), the fully decomposable abelian torsion groups are those which are direct sums of cyclic and quasi-cyclic groups.[2] A criterion for abelian torsion groups to be fully decomposable was kown so far only for the enumerable case, in the form of a theorem of PRÜFER [5] (see Corollary 6 in § 4 of the present paper). In what follows I shall give a criterion for abelian torsion groups of arbitrary power to be fully decomposable (§ 2). This criterion is the generalization of a former result of mine, stating when an abelian p-group of arbitrary power is the direct sum of cyclic groups [2]. Generalizing this result from a different point of view, L. FUCHS obtained recently valuable new results on groups decomposable into the direct sum of cyclic groups [1]. L. FUCHS generalized the criterion of [2], by giving a condition based, instead of the height of group elements, on their "relative order", the latter being closely related to the order of group elements. On the other hand, both the criterion of [2] and that of the present paper are based on the concept of the height of group elements. The criterion in [2] reads as follows: An arbitrary abelian p-group is the direct sum of cyclic groups if and only if the group contains no element of infinite height and there exists a principal system of elements in the group. In the present paper I show that if the concept of elements of infinite height is split in a suitable manner into that of elements of *outer* resp. *inner* infinite height, then — with unchanged definition of principal system (see § 2) — the same criterion holds for an arbitrary abelian p-group to be the direct

[1] The numbers in brackets refer to the Bibliography given at the end of the paper.
[2] For notation and terminology see the next two paragraphs.

sum of cyclic and quasicyclic groups, provided that the term "element of infinite height" is replaced by "element of outer infinite height". (See Theorem in § 2.) Since an abelian torsion group is the direct sum of uniquely determined p-groups, our theorem evidently solves the problem in question for arbitrary torsion groups too. From this criterion we can easily deduce several corollaries (§§ 3, 4) which are partly new and partly well-known theorems. In the proof of Corollary 1, I am indebted for an important idea to L. FUCHS (see [4]). It may be mentioned that the present paper can be read without knowledge of [2]; the proofs are very simple, making use only of the best known fundamental concepts of group theory.

The notations and terminology used are the following. By capital letters we denote groups or some systems of group elements, by the letters $x, a, b, \ldots, g$ group elements, while the other small Latin letters are reserved for rational integers, in particular p for a prime number. The Greek letter ν may range over an arbitrary (not necessarily ordered) set of indices. In what follows, by a group we shall mean always an additively written abelian group with more than one element. A subgroup generated by the elements $a, b, \ldots$ of a group is denoted by $\{a, b, \ldots\}$. A group every element of which is of finite order, is called a *torsion group*. It is well known that a torsion group may be represented as the direct sum of its uniquely determined *primary components*, each of which is a *p-group*, i. e. a group containing only elements of p-power order. We denote the order of a group element a by $O(a)$. The *height* in G of an element a of the p-group G is defined as follows. An element $a \neq 0$ of the p-group G is said to have the height $h = H(a)$ if the equation $p^n x = a$ is solvable in G for $n \leqq h$, but not for $n > h$. We define $H(a) = \infty$ if $p^n x = a$ has a solution $x \in G$ for each natural number n. We emphasize that $H(a)$ is defined only for $a \neq 0$.[3] An important rôle is played in our investigations by the concepts of elements of inner resp. outer infinite height. We say that an element a with $H(a) = \infty$ of the p-group G is an *element of inner infinite height* if $p^t x = a$ has a solution $x \in G$ of infinite height for each natural number t. In the contrary case, when there exists a t for which $p^t x = a$ admits only solutions $x \in G$ of finite height, we call a an *element of outer infinite height*. We remark that if G is the direct sum of its subgroups $B_1, B_2, \ldots$ and $g = b_1 + b_2 + \ldots$ $(b_\nu \in B_\nu)$, then evidently $H(g) \leqq H(b_\nu)$ $(\nu = 1, 2, \ldots)$ holds.

The elements $a_1, \ldots, a_n$ of the group G are called *independent* if $r_1 a_1 + \ldots + r_n a_n = 0$ implies $r_1 a_1 = \ldots = r_n a_n = 0$. An arbitrary set S of elements of G is independent, if every finite subset of S is independent. The independence so defined is therefore a property of finite character and, con-

[3] By the *height* of an element g of G we always mean the number $H(g)$ defined above, i. e. the height refers always to the whole group G, even when an element is, for the moment, considered as an element of some subgroup of G.