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PÓLYA'S THEOREM BY SCHNEIDER'S METHOD

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Dedicated to Professor Th. Schneider on his 65th birthday

A well known theorem of G. Pólya states that 2^z is the smallest transcendental entire function with integral values at all positive integral points z; more precisely, if f is an entire function satisfying $f(n)\in\mathbf{Z}$ for all $n\in\mathbf{N}$, and

$$\limsup_{R\to\infty}\frac{1}{R}\operatorname{Log}|f|_R<\operatorname{Log}2, \tag{1}$$

(where $|f|_R=\sup\limits_{|z|=R}|f(z)|$), then f is a polynomial.

We give here a new proof of this theorem, with a somewhat worse constant in place of Log 2, but which allows some further generalisations.

Notations. We denote by $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{C}$ the non-negative rational integers, the rational integers, the rational numbers and the complex numbers, respectively. When α is an algebraic number, we denote by $s(\alpha)=\max\{\operatorname{Log}|\overline{\alpha}|, \operatorname{Log}d(\alpha)\}$ the size of α (see for instance [3], § 1.2). For $R>0$, B_R is the set $\{z\in\mathbf{C};\ |z|\leqq R\}$. Finally, when $h\in\mathbf{N}$ and $z\in\mathbf{C}$, we define $\binom{z}{h}$ by

$$\binom{z}{h}=\frac{z(z-1)\dots(z-h+1)}{h!}.$$

We shall use only the trivial bounds

$$\left|\binom{z}{h}\right|\leqq 2^{H+R}\quad\text{and}\quad\left|\binom{z}{h}\right|\leqq e^H\left(\frac{R}{H}+1\right)^H$$

for $|z|\leqq R$ and $1\leqq h\leqq H$.

The main result of this paper is the following.

THEOREM 1. *Let K be a number field, and γ_0, γ_1 two positive real numbers. Then there exists an effectively computable number C, depending only on γ_0, γ_1 and $[K:\mathbf{Q}]$, with the following property:*

Let S be a subset of $\mathbf{Z}$, with Card $S\cap B_R\geqq\gamma_0 R$ *for all sufficiently large R; let f, g be two entire functions satisfying*

$$g(n)\neq 0\quad\textit{and}\quad\frac{f(n)}{g(n)}\in K\quad\textit{for all}\quad n\in S,$$

such that for all sufficiently large R,

$$\max_{n\in S\cap B_R} \operatorname{Log}\left\{\frac{1}{|g(n)|};\ s\left(\frac{f(n)}{g(n)}\right)\right\} \leqq \gamma_1 R,$$

and

$$\max\{\operatorname{Log}|g|_R;\ \operatorname{Log}|f|_R\} \leqq R/C.$$

Then f/g is a rational function.

We obtain Pólya's theorem (with the constant Log 2 in (1) replaced by $1/C$) by setting

$$S=\mathbf{N};\ \gamma_0=\gamma_1=1;\ g=1;\ K=\mathbf{Q}.$$

(When $m\in\mathbf{Z}$, then $s(m)=\operatorname{Log}|m|$). A computation[1] of C by the present method leads to $C=283$, and it is an interesting problem to obtain by this way the best possible constant $\frac{1}{\operatorname{Log} 2}=1.44\ldots$.

PROOF OF THEOREM 1. Let k_0 be an integer with $k_0>2\delta/\gamma_0$, where $\delta=[K:\mathbf{Q}]$, and let h_0 be a real number with $2\delta/k_0<h_0<\gamma_0$ (for instance $k_0=[2\delta/\gamma_0]+1$, $h_0=(\gamma_0/2)+(\delta/k_0)$). Let N be a sufficiently large integer; c_1, c_2, c_3 will denote positive constants which are effectively (and easily) computable in terms of $\gamma_0, \gamma_1, \delta$ (and h_0, k_0).

First step. We construct rational integers

$$a_{h,k}\quad (0\leqq h<h_0N;\ 0\leqq k\leqq k_0-1),$$

of absolute value less than $\exp(c_1N)$, *not all zero, such that the meromorphic function*

$$F(z)=\sum_{0\leqq h<h_0N}\ \sum_{0\leqq k<k_0} a_{h,k}\binom{z}{h}\left(\frac{f(z)}{g(z)}\right)^k$$

satisfies

$$F(n)=0 \quad \textit{for all} \quad n\in S\cap B_N.$$

We have to solve a system of at most $2N+1$ linear equations, with at least h_0k_0N unknowns, and with coefficients in K; for $n\in S\cap B_N$, the numbers

$$\binom{n}{h}\left(\frac{f(n)}{g(n)}\right)^k \quad (0\leqq h<h_0N;\ 0\leqq k<k_0)$$

have a common denominator bounded by γ_1k_0N, and a size bounded by $(h_0+1+k_0\gamma_1)N$. Hence Lemma 1.3.1 of [3] gives a non trivial solution $a_{h,k}$ with $\operatorname{Log}\max_{h,k}|a_{h,k}|<c_1N$.

Second step. For $m\in S$, either $F(m)=0$, or $\operatorname{Log}|F(m)|\geqq -c_2|m|$.

The denominator of $F(m)$ is bounded by $\gamma_1k_0|m|$, and the size of $F(m)$ is bounded by

$$\operatorname{Log}[k_0(h_0N+1)]+c_1N+(|m|+h_0N)\operatorname{Log}2+\gamma_1k_0|m|.$$

[1] Made by A. Escassut and M. Mignotte.