Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 33 (3—4), (1979), 307—316.

ON FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY MARTINGALE RANDOM FIELDS

By

A. K. BASU (Sudbury) and C. C. Y. DOREA (Brasilia)

0. Introduction

The term random field is often used to denote a collection of random variables with a parameter space which is a subset of the q-dimensional Euclidean space R^q. Stationary random fields are of great practical importance and hence also of theoretical interest. Examples of random fields occur in biological investigations concerning the distribution of plants or animals over a given area, when $q=2$ and $\mathbf{t}=(t_1, t_2)$ is a point of the area. In problems involving propagation of electromagnetic waves through random media the natural parameter space is a subset of R^4, representing space and time. Further important examples occur in the theory of turbulance where, for example, one may consider the case $q=4$ and $\mathbf{t}$ is a point in space-time, while $\xi_1(\mathbf{t}), \xi_2(\mathbf{t}), \xi_3(\mathbf{t})$ are the velocity components of a turbulent fluid at the point $\mathbf{t}$. Multiparameter stochastic process (the so-called random field) plays a prominent role in weak convergence of empirical process to Kiefer process (a two-dimensional Brownian bridge), Brownian sheets, and sample spacings. In this paper we extend the concept of martingale to random fields and obtain a functional central limit theorem for such random fields. An important example of martingales with a partially ordered parameter set is the following generalization of Wiener process. Let $\mathscr{A}^q$ be the family of all Borel sets in R^q having finite Lebesgue measure. Let $\{X_A, A\in\mathscr{A}^q\}$ be a real Gaussian additive random set function with $E(X_A)=0$, $E(X_A X_B)=m(A\cap B)$ where m denotes the Lebesgue measure. Intuitively, X_A can be thought of as the integral over A of a Gaussian White noise. Such integral of Gaussian White noise has extensively been used by Physicists and engineers.

1. Martingale random fields

Martingales with a partially ordered parameter have been considered by Cairoli [4], Wong and Zakai [10], and recently by Shorack and Smythe [8].

Cairoli's definition of martingale is through product type of probability spaces which is very complicated and of limited scope. Wong and Zakai's approach through increasing path is not suitable for weak convergence. Shorack and Smythe's [8] definition is stronger than our definition if $q>1$. The definition of martingale field given in § 3 is natural for stationary processes which is our primary interest.

2. Notation

Let Z^q denote the set of all q-tuples of integers ($q \geqq 1$, a positive integer). The points in Z^q will be denoted by $\mathbf{m}, \mathbf{n}$, etc., or sometime, when necessary, more explicitly by $(m_1, m_2, \ldots, m_q)$, $(n_1, n_2, \ldots, n_q)$, etc. Z^q is partially ordered by stipulating $\mathbf{m} \leqq \mathbf{n}$ iff $m_i \leqq n_i$ for each $i, 1 \leqq i \leqq q$. We write 0 and 1 for points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^q, respectively.

Let $\{\xi_{\mathbf{n}} : \mathbf{n} \in Z^q\}$ be a random field, i.e., a collection of random variables indexed by time-set Z^q. The random field is said to be stationary if for each finite subset S of Z^q, and each $\mathbf{m} \in Z^q$, the joint distribution of $\{\xi_{\mathbf{n}+\mathbf{m}} : \mathbf{n} \in S\}$ is the same as that of $\{\xi_{\mathbf{n}} : \mathbf{n} \in S\}$. Here $\mathbf{n}+\mathbf{m}$ is the usual coordinatewise sum.

For each $\mathbf{n}(\mathbf{n} \geqq \mathbf{1})$, let $F_{\mathbf{n}}$ be the σ-field generated by $\{\xi_{\mathbf{p}} : p_j \geqq 1$ for $j=1, \ldots, q$ and for at least one i $1 \leqq p_i \leqq n_i\}$. Note that $F_{\mathbf{n}}$ is the σ-field generated by $\{\xi_{\mathbf{p}} : \mathbf{p} \not> \mathbf{n}\}$. For $\mathbf{n} \geqq \mathbf{1}$, define the partial sum $S_{\mathbf{n}} = \sum_{\mathbf{1} \leqq \mathbf{j} \leqq \mathbf{n}} \xi_{\mathbf{j}}$. Whenever convenient we will extend the domain of $S_{\mathbf{n}}$ to include indices some of whose coordinates may be zero. In such cases we define $S_{\mathbf{n}}$ to be zero.

Let T be the closed unit interval $[0, 1]$ and T^q the q-fold Cartesian product of T. Let C^q be the set of all continuous functions on T^q with the uniform metric and, as in BICKEL and WICHURA [1], let us denote by D^q the Skorohod function space on T^q. All the properties of D^q that we need can be found in BICKEL and WICHURA [1].

If $\mathbf{n} = (n_1, n_2, \ldots, n_q)$, let $|\mathbf{n}|$ stand for the product $n_1 n_2 \ldots n_q$. Define $|\mathbf{t}|$ similarly for $\mathbf{t} \in T^q$. In this paper the limit $\mathbf{n} \to \infty$ will mean $\min_{1 \leqq i \leqq q} n_i \to \infty$. On T^q as well as Z^q we use the maximum norm, i.e., if $\mathbf{t} \in T^q$ or $\mathbf{n} \in Z^q$, then $\|\mathbf{t}\| = \max_{1 \leqq j \leqq q} |t_j|$ and $\|\mathbf{n}\| = \max_{1 \leqq j \leqq q} |n_j|$.

If $E\{\xi_{\mathbf{n}}^2\} = \sigma^2 > 0$ for all $\mathbf{n} \geqq \mathbf{1}$, we define for $\mathbf{t} \in T^q$ and $\mathbf{n} \geqq \mathbf{1}$,

$$X_{\mathbf{n}}(t) = (\sigma^2 |\mathbf{n}|)^{-1/2} \times S_{[n_1 t_1], \ldots, [n_q t_q]},$$

where $[\cdot]$ is the usual greatest integer function. The stochastic process $X_{\mathbf{n}}$ has sample paths in D^q.

3. The central limit and related theorems

The main theorem of this paper is

THEOREM 1. *Let $\{\xi_{\mathbf{n}} : \mathbf{n} \geqq \mathbf{1}\}$ be a stationary, ergodic random field for which*

(1) $E(\xi_{\mathbf{n}} \| F_{\mathbf{m}}) = 0$ *whenever* $\mathbf{m} < \mathbf{n}$ *with probability* 1

and for which $E\{\xi_{\mathbf{n}}^2\} = \sigma^2$ is positive and finite. Then the net $\{X_{\mathbf{n}} : \mathbf{n} \geqq \mathbf{1}\}$ of stochastic processes converges weakly, in D^q, to the q-parameter Wiener process.

REMARK. There is no loss of generality in working with the random field $\{\xi_{\mathbf{n}} : \mathbf{n} \in Z^q\}$ since given a random field with "one-sided" time set, we can construct a new random field with time set all of Z^q and with the same finite-dimensional distributions (cf. M. ROSENBLATT [7] in the case $q=2$). Now if we let $F_{\mathbf{n}}^*$ be the