Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 34 (1—2), (1979), 129—139.

MARTINGALE RANDOM CENTRAL LIMIT THEOREMS

By
Z. RYCHLIK (Lublin)

1. Introduction. Let $\{S_n, \mathscr{F}, n \geqq 1\}$ be a martingale on the probability space $(\Omega, \mathscr{A}, P)$, with $S_0=0$, and $X_n=S_n-S_{n-1}$, $n \geqq 1$. $\mathscr{F}_0$ need not be the trivial σ-field $\{\emptyset, \Omega\}$. Let $\varphi_j(t)=E(\exp\{itX_j\}|\mathscr{F}_{j-1})$, and let $\sigma_j^2=E(X_j^2|\mathscr{F}_{j-1})$, $s_n^2=$ $=\sum_{j=1}^{n}\sigma_j^2$ for $n=1, 2, \ldots$.

Now let $\{N_n, n \geqq 1\}$ be a sequence of positive integer-valued random variables defined on the same probability space $(\Omega, \mathscr{A}, P)$. Let us denote

$$S_{N_n} = X_1+X_2+\ldots+X_{N_n}, \quad V_n^2 = \sum_{j=1}^{N_n}\sigma_j^2, \quad f_n(t) = \prod_{j=1}^{N_n}\varphi_j(t/B_n), \quad b_n = (\max_{k \leqq N_n}\sigma_k^2)/B_n^2,$$

where $B_n^2=E(X_1^2+X_2^2+\ldots+X_{N_n}^2)$. Throughout the paper B_n is assumed to be finite for all $n \geqq 1$.

In what follows $C=C[0, 1]$ denotes the space of real-valued, continuous functions on [0, 1] and $\mathscr{G}$ denotes a σ-field of Borel sets generated by the open sets of uniform topology. By W we will denote the Wiener measure on $(C, \mathscr{C})$ with the corresponding Wiener process $\{W(t)\colon 0 \leqq t \leqq 1\}$, (cf. [1], Sec. 9).

Let $Y_n(t)$, $0 \leqq t \leqq 1$, be the random function defined as follows:

$$(1) \qquad Y_n(t) = S_k/V_n + X_{k+1}(tV_n^2-s_k^2)/V_n\sigma_{k+1}^2$$

for $0 \leqq t \leqq 1$ and $s_k^2 \leqq tV_n^2 \leqq s_{k+1}^2$, $k=0, 1, 2, \ldots, N_n-1$, where $s_0^2=0$. It is obvious that $Y_n(t)$ is continuous with probability one, being composed of straight line segments joining the points $(s_k^2/V_n^2,\ S_k/V_n)$, $k=0, 1, 2, \ldots, N_n$. Thus there is a measure P_n in the space $(C, \mathscr{C})$, according to which the stochastic process $\{Y_n(t),\ 0 \leqq t \leqq 1\}$ is distributed.

In this paper we use an approach developed by BROWN [2], to generate random central limit theorems for martingales. Section 2 defines the random Lindeberg condition for martingales and gives its several equivalent forms. Theorems 1 and 2 generalize Lindeberg—Feller's central limit theorem to random sums. Section 3 contains an invariance principle for a certain class of martingales. From this result we obtain some new limit theorems concerning of sums with random indices. The results obtained are generalizations of that of given in [5—7].

Throughout, we use the notations $X_+=\max(0, X)$ and $X_-=\max(0, -X)$, while Re z is used to denote the real part of z. $\Phi(x)$ denotes the standard normal distribution function, and the various kinds of convergence, in L_p norm, in probability, and weak (in distribution) are denoted by $\xrightarrow{L_p}$, $\xrightarrow{P}$ and $\xrightarrow{\mathscr{D}}$, respectively.

2. Random central limit theorems. Throughout the paper we say that the random Lindeberg condition is satisfied if, for all $\varepsilon>0$,

$$L(n,\varepsilon)=B_n^{-2}E\left\{\sum_{i=1}^{N_n}X_i^2I(|X_i|\geqq\varepsilon B_n)\right\}\to 0 \quad\text{as}\quad n\to\infty, \tag{2}$$

where $I(A)$ denotes the indicator function of the set A. Furthermore, we consider martingales for which

$$V_n^2/B_n^2\xrightarrow{P}1 \quad\text{as}\quad n\to\infty. \tag{3}$$

First we shall prove the following

LEMMA 1. *If N_n is stopping time with respect to $\{\mathscr{F}_i,\ i\geqq 0\}$, then* (3) *is equivalent to*

$$V_n^2/B_n^2\xrightarrow{L_1}1 \quad\textit{as}\quad n\to\infty. \tag{4}$$

PROOF. It is obvious that (4) implies (3). On the other hand, for every $\varepsilon>0$,

$$E(V_n^2/B_n^2-1)_-\leqq\varepsilon+P(|V_n^2B_n^{-2}-1|\geqq\varepsilon).$$

Thus, by (3), $\lim_{n\to\infty}E(V_n^2/B_n^2-1)_-=0$. Furthermore,

$$EV_n^2=\sum_{k=1}^{\infty}\int_{[N_n=k]}\sum_{j=1}^{k}\sigma_j^2\,dP=\sum_{k=1}^{\infty}\int_{[N_n\geqq k]}E(X_k^2|\mathscr{F}_{k-1})\,dP=$$

$$=\sum_{k=1}^{\infty}\int_{[N_n\geqq k]}X_k^2\,dP=\sum_{k=1}^{\infty}\int_{[N_n=k]}\sum_{i=1}^{k}X_i^2\,dP=E(X_1^2+\ldots+X_{N_n}^2)=B_n^2.$$

Thus $E(V_n^2/B_n^2-1)=0$, which implies (4) since $E(V_n^2/B_n^2-1)_+=E(V_n^2/B_n^2-1)\to 0$ as $n\to\infty$.

Let us put

$$g(n,\varepsilon)=V_n^{-2}\sum_{j=1}^{N_n}E\big(X_j^2I(|X_j|\geqq\varepsilon B_n)|\mathscr{F}_{j-1}\big),\qquad G(n,\varepsilon)=V_n^2B_n^{-2}g(n,\varepsilon),$$

$$h(n,\varepsilon)=V_n^{-2}\sum_{j=1}^{N_n}E\{X_j^2U(|X_j|B_n^{-1}\varepsilon^{-1})|\mathscr{F}_{j-1}\},\qquad H(n,\varepsilon)=V_n^2B_n^{-2}h(n,\varepsilon),$$

where $U(x)$ is any continuous nonnegative function of bounded variation on $[0,\infty)$ for which $U(0)=0$ and $U(x)\to\text{const.}\,(>0)$ as $x\to\infty$.

LEMMA 2. *Assume that for every n, N_n is a stopping time with respect to $\{\mathscr{F}_i,\ i\geqq 0\}$. Then, under the condition* (3), *or alternatively* (4), *the random Lindeberg condition is equivalent to the convergence to zero as $n\to\infty$ of $g(n,\varepsilon)$, $G(n,\varepsilon)$, $h(n,\varepsilon)$ or $H(n,\varepsilon)$, for all $\varepsilon>0$, either in probability or in L_1.*

PROOF. It suffices to show the mutual equivalence of convergences in probability of g, G, h and H, then to show that each such convergence in probability implies a corresponding convergence in L_1 norm since the convergence in L_1 is stronger than convergence in probability.