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SYMMETRIC MONOTONICITY

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§ 0. Introduction

A real valued function f defined on the real line R is said to be *nondecreasing at* x if there exists a positive number δ_x such that

$$f(x-h) \leqq f(x) \leqq f(x+h) \quad \text{for} \quad 0 < h < \delta_x.$$

The function f is said to be *symmetrically nondecreasing at* x if there exists a positive number δ_x such that

$$f(x-h) \leqq f(x+h) \quad \text{for} \quad 0 < h < \delta_x.$$

We set

$$\mathscr{I} = \{x\colon f \text{ is nondecreasing at } x\}$$

and

$$\mathscr{I}^s = \{x\colon f \text{ is symmetrically nondecreasing at } x\}.$$

The purpose of this paper is to determine the possible size of the set $\mathscr{I}^s - \mathscr{I}$ for various classes of functions. In § 1 we prove that this set is of Lebesgue measure zero for all measurable functions and of the first Baire category for those having the Denjoy property. In § 2 we present several examples to indicate the sharpness of these results.

We shall use the symbol $|E|(|E|^*)$ to denote the Lebesgue measure (outer measure) of a subset E of R.

§ 1. Functions for which $\mathscr{I}^s - \mathscr{I}$ is small

In this section we shall find it convenient to use the decomposition $\mathscr{I}^s = \bigcup_{n=1}^{\infty} S_n$ where $S_n = \{x\colon f(x-h) \leqq f(x+h) \text{ for } 0<h<1/n\}$. We begin with a lemma from which it will readily follow that $\mathscr{I}^s - \mathscr{I}$ is small in the measure-theoretic sense for measurable functions (cf. [3], pp. 217—219).

LEMMA 1. *If $f\colon R \to R$ is measurable and* 0 *is a point of outer density of the set S_n, then there is a positive number δ such that $f(a) \leqq f(b)$ whenever $-\delta < a < 0 < b < \delta$.*

PROOF. Choose a number $\delta \in (0, 1/n)$ such that, for each closed interval I containing 0 and having length less than δ,

$$(1) \qquad |S_n \cap I|^* > 7\,|I|/8.$$

Now, assume that $f(a)>f(b)$ for some pair of points a and b satisfying $-\delta<a<0<b<\delta$. Furthermore, assume $|a|\leqq b$. (The proof in the case when $|a|>b$ is obtained by interchanging the roles of a and b in the ensuing arguments.)

It is a consequence of (1) that $|S_n\cap[b/2, 3b/4]|^*>b/8$. Thus, if we set

$$F=\{x\in[0, b/2]:\ (x+b)/2\in S_n\},$$

then $|F|^*>b/4$ and

$$F\subset G\equiv\{x\in[0, b/2]:\ f(x)<f(a)\}.$$

Since f is measurable, it follows that $|G|>b/4$. Now set $P=\{(a+x)/2:\ x\in G\}$, and let I_0 be the smallest closed interval containing 0 and the set P. It is evident that $|P|>b/8$ and $|I_0|\leqq b/2$; hence, $|P\cap I_0|=|P|>|I_0|/4$. But, since $P\cap S_n=\emptyset$, this contradicts (1) and the proof is complete.

THEOREM 1. *If* f: $R\to R$ *is measurable, then* $|\mathscr{I}^s-\mathscr{I}|=0$.

PROOF. From Lemma 1 we see that if f is approximately continuous at a point x_0 of outer density of S_n, then x_0 is necessarily in $\mathscr{I}$. Then since almost every point of S_n is both a point of approximate continuity of f and a point of outer density of S_n, it follows that $|S_n-\mathscr{I}|=0$ for each index n, and the theorem is proved.

We now proceed to establish the smallness of $\mathscr{I}^s-\mathscr{I}$ in the topological sense for functions possessing the Denjoy property. (Here we say that a function f: $R\to R$ has the Denjoy property if for every pair of open intervals I and J the set $I\cap f^{-1}(J)$ is either empty or of positive Lebesgue measure; we note that the class of functions having the Denjoy property contains the class of approximately continuous functions and the class of Baire*1 Darboux functions, recently introduced by R. J. O'MALLEY [5].) First we prove two lemmas which may have interest independent of the theorem they precede.

LEMMA 2. *Let* f: $R\to R$ *have the Denjoy property. If* S_n *is dense in an open interval* I, *then* $I\subset S_n$.

PROOF. Suppose I is not contained in S_n and for notational simplicity assume $0\in I-S_n$. Since $0\notin S_n$ there is a number h satisfying $0<h<1/n$ such that $f(-h)>f(h)$. Choose δ such that $0<\delta<\min\{h, 1/n-h\}$ and $(-\delta, \delta)\subset I$. Employing the Denjoy property of f, we can find two sets A and B of positive measure such that $A\subset(-h-\delta, -h+\delta)$, $B\subset(h-\delta, h+\delta)$ and

$$f(a)>f(b)\quad\text{for all}\quad a\in A,\quad b\in B. \tag{2}$$

Let $C=\{(a+b)/2:\ a\in A, b\in B\}$. Since A and B have positive measure, C must contain an interval (see [2, Theorem B, p. 68]). Furthermore $C\subset(-\delta, \delta)\subset I$, and $C\cap S_n=\emptyset$ because of (2). This contradicts the hypothesis that S_n is dense in I, and the proof is complete.

The next lemma generalizes Theorem 1 of [4] where the same result is proved for continuous functions by different methods. We should also observe that based on the following result, it is easy to verify that all fifteen theorems and corollaries proved in [4] for continuous functions remain true for functions possessing the Denjoy property.