Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 35 (1—2), (1980), 123—128.

OPERATORS ON BANACH SPACES WITH COMPLEMENTED RANGES

By

S. L. CAMPBELL and G. D. FAULKNER (Raleigh)

This paper is concerned with when the closure of the range, $\overline{R(A)}$, of a bounded linear operator A, acting on a Banach space X, is complemented. Several sets of sufficient conditions will be given. Examples will show that some type of assumptions, similar to those made here, are necessary.

1. Introduction. The concept of a generalized inverse has been useful in a variety of settings. The uses of generalized inverses in finite dimensional spaces include solving linear systems, least squares analysis, solving differential equations [5], Markov Chains [5], and so on.

The usefulness of the concept in Banach spaces has been studied in [1], [2], [3], [6], [8], [10], [11], [15]. A 1,775 item bibliography on generalized inverses may be found in [13].

An operator T is called a $(1, 2)$-inverse of A if $ATA=A$, $TAT=T$. ($(1, 2)$-inverses are among the more important generalized inverses, especially in solving consistent linear equations.) An operator A will have a bounded $(1, 2)$-inverse if and only if $\overline{R(A)}$ is complemented and the nullspace of A, $N(A)$, has a complement on which A is bounded below ($\|Ax\| \geqq m\|x\|$, $m>0$). In particular, if A is bounded below, then A has a bounded $(1, 2)$-inverse if and only if $R(A)$ is complemented.

Thus conditions which guarantee that $\overline{R(A)}$ be complemented are not only of obvious theoretical interest but have immediate application to the theory of generalized inverses in Banach spaces.

There are two types of results that deserve mention here. There exist many results, see [7], [17], [18], for examples, which give that $\overline{R(A)}$ is complemented (in fact by $N(A)$) if 0 is a boundary point of the spectrum of A, $\sigma(A)$, and the resolvent of A satisfies an appropriate growth condition. The other sometimes related results try to mimic the result that for an operator on a Hilbert space $R(A)\oplus \oplus R(A^*)=X$. Our results will not be of this type.

We shall denote the set of all bounded linear operators from X into X with $\overline{R(A)}$ complemented by $\tilde{\zeta}(X)$. The closed linear span of a set $S\subseteq X$ is denoted $[S]$. Subspaces are assumed closed. Uniform, strong, and weak limits are denoted by $\Rightarrow$, $\rightarrow$, and $\rightharpoonup$, respectively.

2. Limits of complemented operators. Our first results show that if A is the limit of $A_n \in \tilde{\zeta}(X)$ and the $\overline{R(A_n)}$ and their complements have limits in the appropriate sense, then $A\in \tilde{\zeta}(X)$ under reasonable assumptions. First we need a definition and a technical result.

DEFINITION 1. Let $\{M_n\}$ be a sequence of subspaces of X. If

$$\bigcap_{n\geqq 1}\Big[\bigcup_{k\geqq n} M_k\Big] = \overline{\bigcup_{n\geqq 1}\bigcap_{k\geqq n} M_k} \tag{1}$$

holds, the common value of the two sets is denoted $\lim M_n$.

If $\{M_n\}$ is monotonic, then $\lim M_n$ exists. In fact $\lim M_n=\overline{\bigcup_n M_n}$ for increasing sequences and $\lim M_n=\bigcap_n M_n$ for decreasing sequences. However, $\{M_n\}$ may have a limit without being monotonic. For example, $\lim_n [\{e_1, e_n\}]=[e_1]$ if $\{e_k\}$ is the standard basis in l^p, $1\leqq p<\infty$.

THEOREM 1. *Let X be a reflexive Banach space, M_k, N_k subspaces such that $X=M_k\oplus N_k$, and P_k the projection onto M_k along N_k. That is, $P_kX=M_k$, $(I-P_k)X=N_k$. If $\|P_k\|\leqq K$ for all k and $\lim M_k$, $\lim N_k$ exist, then $X=\lim M_k\oplus\lim N_k$.*

PROOF. Let $z\in X$ with $z=m_k+n_k$, $m_k\in M_k$, $n_k\in N_k$. Since $K\|z\|=K\|m_k+n_k\|\geqq\|m_k\|$, $m_{k_j}\rightharpoonup m$ for some subsequence m_{k_j} of m_k. Since $m_{k_j}\in[\bigcup_{k\geqq n} M_k]$ for all but finitely many terms, and $[\bigcup_{k\geqq n} M_k]$ is weakly closed, it follows that $m\in\lim M_k$. Also we have $z-m_{k_j}=n_{k_j}\rightharpoonup z-m$. Since $P_{k_j}(z-m_{k_j})=0$, we also have $z-m\in\lim_{n\to\infty} N_n$. Thus $\lim M_k+\lim N_k=X$. To see the sum is direct, suppose $z\in\lim M_k\cap\lim N_k$. Then there exists $w_j\in\bigcup_{n\geqq 1}\bigcap_{k\geqq n} M_k$ and $v_j\in\bigcup_{n\geqq 1}\bigcap_{k\geqq n} N_k$ with $w_j\to z$ and $v_j\to z$. Now $w_j\in\bigcap_{k\geqq n_j} M_k$ and $v_j\in\bigcap_{k\geqq m_j} N_k$ for some m_j, n_j. Let $r_j=\max\{m_j, n_j\}$. Then $P_{r_j}(m_j-n_j)=m_j$ and $P_{r_j}(n_j)=0$. But $w_j-v_j\to 0$, so that $\|P_{r_j}(w_j-v_j)\|\leqq K\|w_j-v_j\|\to 0$ and hence $\|w_j\|\to 0$. Since $w_j\to z$ we must have $z=0$. Q.e.d.

THEOREM 2. *Let X be a reflexive Banach space, A_n, A operators on X. Suppose that $A_n\to A$ strongly and $A_n\in\tilde{\zeta}(X)$. Let $M_n=\overline{R(A_n)}$. Suppose there exist projections P_n such that $P_n(X)=M_n$, $(I-P_n)(X)=N_n$, $\|P_n\|\leqq K$, and $M=\lim M_n$ and $N=\lim N_n$ exist with $M\subseteqq\overline{R(A)}$. Then $A\in\tilde{\zeta}(X)$.*

PROOF. By Theorem 1, M is complemented in X and $M\subseteqq\overline{R(A)}$. Suppose now that $y\in R(A)$ i.e. $Ax=y$. It follows that $A_nx\to Ax=y$. But for each n, $A_nx\in M_n$, and thus $A_nx\in[\bigcup_{k\geqq m} M_k]$ for $m\leqq n$. Thus $Ax=y\in[\bigcup_{k\geqq m} M_k]$ for each m, so that $y\in\bigcap_n[\bigcup_{k\geqq n} M_k]=M$. Hence $A\in\tilde{\zeta}(X)$ since $M=\overline{R(A)}$ is complemented. Q.e.d.

The assumptions of Theorem 2 appear quite strong. However, it is very difficult to get a much more general theorem.

EXAMPLE 1. Let A be a bounded below operator in l^p, $1<p<\infty$, $p\neq 2$, such that $R(A)$ is not complemented [16]. Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis for l^p. Define $A_ne_i=Ae_i$ if $i\leqq n$, $A_ne_i=0$ if $i>n$. Then $A_n\to A$. Since $R(A_n)$ is finite dimensional it has a complement N_n. Since $R(A_n)$ is monotonically increasing and the codimension of $R(A_n)$ in $R(A_{n+1})$ is one, it is possible to inductively define the N_n so they are decreasing. That is, $N_n\supseteqq N_{n+1}$. Define P_n by $P_nM_n=M_n$, $P_nN_n=0$. Then $A_n\to A$, $A_n\in\tilde{\zeta}(X)$, $\lim M_n$, $\lim N_n$ exist, $\lim M_n\subseteqq\overline{R(A)}$, and $A\notin\tilde{\zeta}(K)$.