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ON THE STRONG APPROXIMATION OF FOURIER SERIES

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1. Let f be a continuous and 2π-periodic function. Denote $\omega(f;\delta)$ and $s_n(x)=s_n(f;x)$ the modulus of continuity of f and the n-th partial sum of its Fourier series, respectively.

If ω is a modulus of continuity and r is a natural number we define W^rH^ω as the class of all functions f, for which

$$\omega(f^{(r)};\delta)\leqq K_f\omega(\delta)\quad(0\leqq\delta\leqq 2\pi)$$

where K_f is a constant (depending generally on f).

The first result on strong approximation is due to ALEXITS and KRÁLIK [1]. In 1965 LEINDLER [2] proved a very general theorem about the order of strong approximation. This result can be applied to the most important strong means

$$h_n(f,p,\beta;x)=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad(\beta,p>0),$$

$$\tilde h_n(f,p,\beta;x)=h_n(\tilde f,p,\beta;x),$$

$$\sigma_n^\gamma|f,p;x|=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad\left(\gamma,p>0,\ A_n^\gamma=\binom{n+\gamma}{n}\right),$$

and

$$\tilde\sigma_n^\gamma|f,p;x|=\sigma_n^\gamma|\tilde f,p;x|.$$

One can get e.g. the following theorem (see [2]).

THEOREM A. *Let us suppose that* $f^{(r)}\in\operatorname{Lip}\alpha$ $(0<\alpha\leqq1)$ *and* $p>0$. *If* $\beta>(r+\alpha)p$ *then*

$$h_n(f,p,\beta;x)=O(n^{-r-\alpha})$$

and

$$\tilde h_n(f,p,\beta;x)=O(n^{-r-\alpha})$$

while if $\beta=(r+\alpha)p$ *then we have only*

$$h_n(f,p,\beta;x)=O\big(n^{-r-\alpha}(\log n)^{1/p}\big)$$

and

$$\tilde h_n(f,p,\beta;x)=O\big(n^{-r-\alpha}(\log n)^{1/p}\big).$$

Moreover, there are functions f_1 *and* f_2, *so that* $f_i^{(r)}\in\operatorname{Lip}\alpha$ $(i=1,2)$, *but*

$$h_n(f_1,p,\beta;0)\geqq cn^{-r-\alpha}(\log n)^{1/p}\quad(c>0)$$

and

$$\tilde h_n(f_2,p,\beta;0)\geqq cn^{-r-\alpha}(\log n)^{1/p}.$$

LEINDLER [5] proved also that in the case $\alpha=1, \beta=(r+1)p$ the additional assumption $\tilde{f}^{(r)}\in \mathrm{Lip}\,\alpha$ does not improve the above estimation.

In [3] we can find the analogue of Theorem A for the means $\sigma_n^\gamma|f, p; x|$ and $\tilde{\sigma}_n^\gamma|f, p; x|$.

In the first part of the present paper we give the exact approximation order, which can be achieved by the above means, if f is taken from a class $W^r H^\omega$. After that we shall deal with the so-called generalized strong de la Vallée Poussin means.

Let ω be an arbitrary modulus of continuity. We define

$$\omega^*(\delta) = \int_0^\delta \frac{\omega(t)}{t}\,dt.$$

It is possible to see that in the case $\int_0^1 \frac{\omega(t)}{t}\,dt<\infty$, $\omega^*(\delta)$ is between a modulus of continuity and its twofold, so we may regard $\omega^*(\delta)$ as a modulus of continuity.

Let ω_0 be the infimum of those α, for which

$$\sum_{k=0}^{n} 2^{k\alpha}\omega\left(\frac{1}{2^k}\right) \leqq K_\alpha 2^{n\alpha}\omega\left(\frac{1}{2^n}\right) \quad (n = 0, 1, 2, \ldots) \tag{1}$$

is true with a constant K_α.

By Lemma 2 it is clear that if (1) holds for a certain α then it holds for any $\alpha'>\alpha$ and there exists a positive $\varepsilon=\varepsilon(\alpha)$ such that (1) holds for $\alpha-\varepsilon$, too.

Thus by the definition of ω_0 it is clear that (1) holds if and only if $\alpha>\omega_0$.

With the above notations we prove

THEOREM 1. *Let $f\in W^r H^\omega$ and $p>0$. We have*

$$h_n(f;\, p, \beta;\, x) = O(H_{r,\omega}^{p,\beta,n}) \tag{2}$$

where

$$H_{r,\omega}^{p,\beta,n} = \left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}.$$

Moreover if $r>0$ then

$$\tilde{h}_n(f, p, \beta;\, x) = O(H_{r,\omega}^{p,\beta,n}), \tag{3}$$

while if $r=0$ then

$$\tilde{h}_n(f, p, \beta;\, x) = O(H_{0,\omega^*}^{p,\beta,n}) \tag{4}$$

are true.

Furthermore, there are functions f_r $(r=0, 1, \ldots)$ and f_0^* so that $f_r, \tilde{f}_r\in W^r H^\omega$ $(r=0, 1, \ldots)$, $f_0^*\in H^\omega$, and

$$h_n(f_r, p, \beta;\, 0) \geqq cH_{r,\omega}^{p,\beta,n} \tag{5}$$

$$\tilde{h}_n(f_r, p, \beta;\, 0) \geqq cH_{r,\omega}^{p,\beta,n} \tag{6}$$

$$\tilde{h}_n(f_0^*, p, \beta;\, 0) \geqq cH_{0,\omega^*}^{p,\beta,n} \tag{7}$$

are true with a positive constant c independent of n.