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THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR LACUNARY TRIGONOMETRIC SERIES

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1. Introduction. In this note let $\{n_m\}$ be a sequence of positive integers satisfying the gap condition

$$n_{m+1}/n_m > 1+cm^{-\alpha} \quad (c>0 \text{ and } 0 \leqq \alpha \leqq 1/2), \tag{1.1}$$

and $\{a_m\}$ be a sequence of positive numbers such that

$$A_k = \left(2^{-1}\sum_{m=1}^{k} a_m^2\right)^{1/2} \to +\infty \quad \text{and} \quad a_k = o(A_k k^{-\alpha}), \quad \text{as} \quad k \to +\infty. \tag{1.2}$$

Then we put, for any sequence $\{\alpha_m\}$ of real numbers

$$\xi_m(\omega) = a_m \cos 2\pi(n_m\omega+\alpha_m) \quad \text{and} \quad S_k(\omega) = \sum_{m=1}^{k} \xi_m(\omega). \tag{1.3}$$

Consider ξ_m'S as random variables on a probability space $([0, 1), \mathcal{F}, P)$ where $\mathcal{F}$ is the σ-field of all Borel sets on $[0, 1)$ and P is the Lebesgue measure on $\mathcal{F}$. Further we write, for $\omega \in [0, 1)$, $t \in [0, 1]$ and every positive integer k,

$$X_k(t) = X_k(t, \omega) = A_k^{-1} S_m(\omega), \quad \text{if} \quad A_m^2 A_k^{-2} \leqq t < A_{m+1}^2 A_k^{-2}.[1] \tag{1.4}$$

Then $X_k(t)$ is a random element of $(D, \mathcal{D})$ defined on the probability space $([0, 1), \mathcal{F}, P)$ where D is the set of real-valued functions that are right continuous and have left-hand limits and $\mathcal{D}$ is the Skorohod σ-field in D (cf. [2], p. 111).

The purpose of the present paper is to prove the following

THEOREM. *We have* $X_k(t) \Rightarrow X(t)$, *in* $(D, \mathcal{D})$, *as* $k \to +\infty$, *where* $\{X(t), 0 \leqq t \leqq 1\}$ *is the standard Brownian motion.*

In [4] we proved that if μ is a probability measure on $([0, 1), \mathcal{F})$ such that $\mu \ll P$, then for any real number x

$$\lim_{k\to\infty} \mu\{\omega;\, X_k(1, \omega) \leqq x\} = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)\, du. \tag{1.5}$$

Further, in [5] it is proved that if we replace the condition $a_k = o(A_k k^{-\alpha})$ in (1.2) by $a_k = O(A_k k^{-\alpha})$ as $k \to +\infty$, then (1.5) does not necessarily hold for the measure P. Therefore (1.2) is the best possible condition for the functional central limit theorem of lacunary trigonometric series.

[1] We put $A_0 = 0$.

In case $0 \leqq \alpha < 1/2$ and $a_k=1$ for all k I. BERKES [1] proved an almost sure invariance principle for $X_k(t)$.

For the proof of our theorem we approximate $\{S_k(\omega)\}$ by a martingale and then apply a martingale version of the functional central limit theorem due to McLEISH ([3], (3.2)).

2. Preliminaries. Let us put, for each integer k

$$\begin{cases} p(0)=0, \quad p(k)=\max\{m;\, n_m < 2^k\} \\ \Delta_k = \sum_{m=p(k)+1}^{p(k+1)} \xi_m \quad \text{and} \quad B_k = A_{p(k+1)}.\text{[2]} \end{cases} \tag{2.1}$$

Then if $p(k)+1<p(k+1)$, we have, by (1.1)

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1} (1+cm^{-\alpha}) > 1+c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1).$$

Hence we have

$$p(k+1)-p(k) = O(p^\alpha(k)), \quad \text{as} \quad k\to+\infty \tag{2.2}$$

and if $m_k=o(p^{1-\alpha}(k))$ as $k\to+\infty$, then

$$p(k+m_k)/p(k) \to 1, \quad \text{as} \quad k\to+\infty. \tag{2.3}$$

From (1.2) and (2.2), it is easily seen that

$$b_k = \max_{p(k)<m\leqq p(k+1)} |a_m| = o(B_k p^{-\alpha}(k)), \quad \text{as} \quad k\to+\infty.$$

Hence we can take an increasing sequence $\{g(k)\}$ of real numbers such that

$$\begin{cases} g(k) \leqq \min\{p(k), B_k g(k-1)B_{k-1}^{-1}\} \quad \text{for} \quad k>1 \quad \text{and} \\ g(k)\to+\infty, \quad b_k = O(B_k/p^\alpha(k)g(k)), \quad \text{as} \quad k\to+\infty. \end{cases} \tag{2.4}$$

Therefore, we have

$$\begin{cases} \sum_{m=p(k)+1}^{p(k+1)} |a_m| \leqq b_k\{p(k+1)-p(k)\} = O(B_k/g(k)), \\ E\Delta_k^2 \leqq b_k^2\{p(k+1)-p(k)\} = O(B_k^2/p^\alpha(k)g^2(k)), \quad \text{as} \quad k\to+\infty. \end{cases} \tag{2.5}$$

LEMMA 1. *For any given integers k, j, q and h such that*

$$p(j)+1 < h \leqq p(j+1) < p(k)+1 < q \leqq p(k+1)$$

the number of solutions (n_r, n_i) of the equations

$$n_q - n_r = n_h \pm n_i$$

where $p(j)<i<h$ and $p(k)<r<q$, is at most $C2^{j-k}p^\alpha(k)$ where C is a positive constant which does not depend on k, j, q and h.

[2] $\Delta_k=0$ if $p(k)=p(k+1)$.