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ON SOME TRIGONOMETRIC DIOPHANTINE EQUATIONS OF THE TYPE $\sqrt{n} = c_1 \cos \pi d_1 + \ldots + c_\lambda \cos \pi d_\lambda$

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§ 1. Introduction

The object of this paper is to give the complete solution of the equation (*) of the title for $n=2p$ (p odd prime), $3p$ (p prime >3) and pq (p, q distinct primes $\equiv 1 \pmod 4$), where the c_j, d_j are rational and where λ is the minimum integer for which $\sqrt{n}$ can be so expressed. We have already done this for $n=p$ in [2] and the value of λ for the n mentioned above has already been determined in [3].

If S is a rational linear combination of roots of unity then we denote by $|S|$ the number of terms in S and by $v(S)$ the value of S as a complex number. S itself will stand to mean the entire expression for S. We regard the two solutions $\sum_{j=1}^{\lambda} c_j \cos \pi d_j$ and $\sum_{j=1}^{\lambda} c'_j \cos \pi d'_j$ as the same if the sets $\{c_j \cos \pi d_j\}$ and $\{c'_j \cos \pi d'_j\}$ are the same. Any rational term in a solution of (*) will be written as $c \cos \pi/3$.

§ 2.

We prove the following

THEOREM 1. *All the solutions of* (*) *for* $n=2p$ *are*

$$\sqrt{2p} = 2\cos \pi/4 - 4 \sum_{(R/p)=1} \cos (3\pi/4 + 2\pi R/p)$$

and

$$\sqrt{2p} = -2\cos \pi/4 + 4 \sum_{(N/p)=-1} \cos (3\pi/4 + 2\pi N/p),$$

unless $p=3$ *in which case* (*) *has one more solution viz*

$$\sqrt{6} = 2\cos \pi/12 + 2\cos 5\pi/12.$$

PROOF. First let $p \equiv 1 \pmod 4$. Then (*) is

$$\sqrt{2p} = \sum_{j=1}^{(p+1)/2} \frac{1}{2} c_j (\zeta_j + \zeta_j^{-1}) \tag{2.1}$$

(where $\zeta_j = e^{2\pi i/2d_j}$), which is a rational linear combination of roots of unity involving $p+1$ terms. Hence by Theorem 3 of [3] this sum is minimal and by Lemma 1 of [3], $p^2 \nmid d_j$, so we can write $\zeta_j = \eta_j \zeta^{e_j}$ where $p \nmid o(\eta_j)$ and where $\zeta = e^{2\pi i/p}$ $(0 \leqq e_j \leqq p-1)$. (2.1) gives

$$\sqrt{2p} = S_0 + S_1 \zeta + \ldots + S_{p-1} \zeta^{p-1} \tag{2.2}$$

where the orders of all the roots of unity involved in each S_j are prime to p. Further

$$(2.3) \qquad \bar{S}_0 = S_0, \quad \bar{S}_j = S_{p-j}, \quad \sum_{j=1}^{p-1} |S_j| = p+1.$$

Now

$$(2.4) \qquad \sqrt{2p} = \sqrt{2}\cdot\sqrt{p} = (\eta-\eta^3)+\sum_{(R/p)=1} 2(\eta-\eta^3)\zeta^R$$

where $\eta=e^{2\pi i/8}$, so that $\eta-\eta^3=\sqrt{2}$. Comparing (2.2) and (2.4) and using Lemma 1 of [1] we get (for quadratic residues R and non-residues N mod p)

$$(2.5) \qquad v(S_0)-\sqrt{2} = v(S_R)-2\sqrt{2} = v(S_N).$$

Case 1. $v(S_{R_0})=0$ *for some* R_0 *(and so for all R)*. Then by (2.5), $v(S_N) = -2\sqrt{2}$, $v(S_0) = -\sqrt{2}$ and so S_0 and each S_N involves at least 2 roots of unity i.e. $|S_0|\geqq 2$, $|S_N|\geqq 2$. Hence by (2.3) $|S_0|=|S_N|=2$, $|S_R|=0$. It follows that $S_N= -2(\eta-\eta^3)$, $S_0=-(\eta-\eta^3)$ since the only way of expressing $\sqrt{2}$ is this, by Lemma 4, part (iii) of [3]. Hence (2.2) now gives the second solution of the theorem.

Case 2. $v(S_{N_0})=0$ *for some* N_0. Then a similar argument gives the first solution of the theorem.

Case 3. $v(S_0)=0$. Then each S_R and each S_N contains at least 2 terms (as in case 1). Totgether they make $2(p-1)$ terms and this is $>p+1$ since $p>3$ (as $p\equiv 1 \pmod 4$). So this case is impossible by (2.3).

Case 4. *None of* $v(S_j)=0$. Then by (2.3) each S_R and each S_N contains exactly one term and S_0 contains 2 terms and so $v(S_R)=S_R$ and $v(S_N)=S_N$. It follows by (2.5) that *either* $S_R=2\eta$ (all R), $S_N=2\eta^3$ (all N) and $S_0=(\eta+\eta^3)$ *or* $S_R=-2\eta^3$ (all R), $S_N=-2\eta$ (all N) and $S_0=-(\eta+\eta^3)$. But S_0 has to make up one cosine since $S_j\zeta^j$ and $S_{p-j}\zeta^{p-j}$ combine to make one $(j=1, 2, \ldots, (p-1)/2)$ and that is not possible since $S_0=-(\eta+\eta^3) = -i\sqrt{2}$ is not real. This case is thus impossible too.

Next let $p\equiv 3 \pmod 4$. Here the equation corresponding to (2.4) is $\sqrt{2p}= -(\eta+\eta^3)\{1+2\sum_{(R/p)=1}\zeta^R\}$ (on writing $\sqrt{2p}=-\sqrt{-2}\cdot\sqrt{-p}$). Proceeding as for the case $p\equiv 1 \pmod 4$ we arrive at the following relations:

$$v(S_0)+(\eta+\eta^3) = v(S_R)+2(\eta+\eta^3) = v(S_N).$$

Here if $v(S_R)=0$ for any one (and so for all) R then $v(S_0)=\eta+\eta^3$ which is not real and so is not a cosine.This case is thus impossible. Similarly the case $v(S_N)=0$ is impossible. If $v(S_0)=0$ we proceed as in case 3 of $p\equiv 1 \pmod 4$ and see that we get no solution unless $p=3$ in which case $S_1 = -(\eta+\eta^3)$, $S_2=\eta+\eta^3$ giving the extra solution for $\sqrt{6}$ mentioned in the theorem. Finally if none of the $v(S_j)$ equal 0, then as before there are just two possibilities viz.

$$S_R = -2\eta, \quad S_N = 2\eta^3, \quad S_0 = -\eta+\eta^3 \quad \text{and} \quad S_R = -2\eta^3, \quad S_N = 2\eta, \quad S_0 = \eta-\eta^3.$$

These give the two stated solutions.