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THE DISTRIBUTION OF THE CHARACTER DEGREES OF THE SYMMETRIC p-GROUPS

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1. Introduction. For *fixed* prime p, let P_n denote the Sylow p-subgroup of the symmetric group S_{p^n} on p^n letters. We call these groups P_n $(n=1, 2, \dots)$ "symmetric p-groups". We have obviously

$$|P_n| = p^{1+p+\dots+p^{n-1}}. \tag{1.1}$$

The thorough study of the symmetric p-groups was initiated by L. A. Kaloujnine [6], [7].

In this paper we shall investigate the degrees of the irreducible characters (i.e., the dimensions of the pairwise non-equivalent irreducible representations (over the complex field)) of P_n from the point of view of the statistical (or probabilistic) group theory. Since the character degrees divide the order of the group, in our case each degree is a power of p, therefore we rather consider $\log_p \chi(1)$. The probability measure will simply be the proportion of a subset of irreducible characters. So the number h_n of conjugacy classes of P_n will also play an important role. *Our main purpose is to prove that* $\log_p \chi(1)$ *shows an asymptotically normal distribution.*

In a previous paper (P. P. Pálfy and M. Szalay [10]) we investigated the distribution of the orders of the elements of P_n. This work was inspired by the celebrated series of papers [4] by P. Erdős and P. Turán on the statistical group theory dealing with the distribution of the orders of the elements of S_n and related problems. (For a simpler proof of their main distribution theorem, see J. D. Bovey [2]. J. Dénes, P. Erdős and P. Turán [3] proved an analogous distribution theorem for the alternating group A_n.)

For the dimensions of the complex irreducible representations of S_n, one has the trivial *upper* bound

$$\sqrt{n!} = \exp\left\{\frac{1}{2} n \log n - \frac{1}{2} n + O(\log n)\right\}.$$

Dealing with the value distribution of the complex irreducible characters of S_n, P. Turán [13] (see also [14]) remarked that the *maximal* dimension is

$$\exp\left\{\frac{1}{2} n \log n - \frac{1}{2} n + O\left(\sqrt{n}\right)\right\} \tag{1.2}$$

owing to the relatively small class number of S_n which is $p(n)$, the number of partitions of n. M. P. Schützenberger called afterwards his attention to the interest of the question what can be said on the distribution of the dimensions. According to the first result (M. Szalay [11]), the dimensions of *almost all* complex irreducible representations of S_n are of the form

$$\exp\left\{\frac{1}{2} n \log n - O(n \log\log n)\right\}. \tag{1.3}$$

By means of the statistical investigation of partitions (M. Szalay and P. Turán [12]), this was improved to

$$\exp\left\{\frac{1}{2} n \log n - \left(\frac{1}{2}+A\right) n + O(n^{7/8}\log^4 n)\right\} \tag{1.4}$$

with a positive constant A. As P. Erdős remarked, this *cannot* be improved to

$$\exp\{g(n)+O(n^{1/2}\log^{-1} n)\}.$$

Another natural probability measure for the irreducible representations of a group G is the Plancherel measure, where the probability of the character χ is $\chi^2(1)/|G|$. Results concerning the Plancherel measure of S_n have been obtained by A. M. Veršik and S. V. Kerov [15].

In what follows we are going to investigate the *distribution* of the dimensions for the symmetric p-group P_n. As to the *maximal* dimension, we shall show that, e.g., for $p>2$ and $n>1$, it is

$$p^{1+p+\ldots+p^{n-2}}$$

(see Proposition 1). Our main purpose is to prove the following distribution theorem by the moment method. (Our random field consists of all possible choices of complex irreducible characters of P_n with equal probabilities.)

THEOREM. *There exist positive constants* $\bar{\alpha}=\bar{\alpha}(p)$, $\bar{\beta}=\bar{\beta}(p)$ *with*

$$1+\frac{1}{p-1}-\frac{1}{p^{p-3}} < \bar{\alpha} < 1+\frac{1}{p-1} \tag{1.5}$$

and

$$1-\frac{1}{p^{p-3}} < \bar{\beta} < 1-\frac{1}{p^{p-3}}+\frac{2}{p^{p-2}} \tag{1.6}$$

such that, for a randomly chosen complex irreducible character χ *of* P_n, *we have*

$$\lim_{n\to\infty} \operatorname{Prob}\left(\frac{\log_p \chi(1)-\bar{\alpha}p^{n-2}}{\bar{\beta}p^{(n-p+1)/2}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt. \tag{1.7}$$

Another result worth mentioning here is the following formula for the class number h_n of P_n (see (3.4)—(3.5)):

$$h_n = \left[p^{\frac{\bar{\gamma}+1}{p-1}}\cdot|P_n|^{\bar{\gamma}}\right] - \begin{cases} 0 & \text{if } p>2 \\ 1 & \text{if } p=2 \end{cases}$$

where $\bar{\gamma}=\bar{\gamma}(p)$ is a constant with $0<\bar{\gamma}<1$ (cf. (3.6)).

The constants $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are given by infinite series containing the class numbers h_n $(n=1, 2, \ldots)$ (cf. (4.6) and (4.11); (4.9) and (4.11); (3.3) and (3.5)). Fortunately h_n grows rapidly, hence the convergence is fast. Here we give the values of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ for the smallest primes.