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ARITHMETICAL PROPERTIES OF PERMUTATIONS OF INTEGERS

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For the finite case let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$ and for the infinite case let $a_1, a_2, \ldots, a_i, \ldots$ be a permutation of all positive integers.

Some problems and results concerning such permutations and related questions can be found in [2] (see in particular p. 94). In [3] the density of the sums a_i+a_{i+1} is estimated from several points of view.

In the present paper we shall investigate the least common multiple and the greatest common divisor of two subsequent elements. First we deal with the least common multiple. For the identical permutation we have $[a_i, a_{i+1}]=i(i+1)$. We show that for suitable other permutations this value becomes considerably smaller.

First we consider the finite case

THEOREM 1. *We have*

$$\min \max_{1\leq i\leq n-1} [a_i, a_{i+1}] = \bigl(1+o(1)\bigr)\frac{n^2}{4\log n} \tag{1}$$

where the minimum is to be taken for all permutations $a_1, a_2, \ldots, a_n$.

One might think that the main reason for not being able to get a smaller value lies in the presence of the large primes (see also the proof). Theorem 2 shows that this is only partly true.

THEOREM 2. *Omit arbitrarily* $g(n)=o(n)$ *numbers from* $1, 2, \ldots, n$ *and form a permutation of the remaining ones.*

Then for any fix $\varepsilon>0$, *and n large enough we have*

$$\min \max_{1\leq i\leq n-g(n)-1} [a_i, a_{i+1}] > n^{2-\varepsilon}. \tag{2}$$

On the other hand, for any $\varepsilon(n)\to 0$ *we have with a suitable* $g(n)=o(n)$

$$\min \max_{1\leq i\leq n-g(n)-1} [a_i, a_{i+1}] < n^{2-\varepsilon(n)}. \tag{3}$$

An equivalent form of Theorem 2 is: $\dfrac{\log\{\min \max_{1\leq i\leq n-g(n)-1} [a_i, a_{i+1}]\}}{\log n}$ must tend to 2 for any $g(n)=o(n)$, but it can do this from below arbitrarily slowly for suitable $g(n)=o(n)$.

In the infinite case we obtain a much smaller upper bound:

THEOREM 3. *We can construct an infinite permutation satisfying*

$$[a_i, a_{i+1}] < ie^{c\sqrt{\log i}\log\log i} \tag{4}$$

for all i.

In the opposite direction we can prove only a very poor result:

THEOREM 4. *For any permutation*

$$\limsup_i \frac{[a_i, a_{i+1}]}{i} \geqq \frac{1}{1-\log 2} \sim 3{,}26. \tag{5}$$

Very probably this lim sup must be infinite, and one can expect an even sharper rate of growth.

Concerning the greatest common divisor only the infinite case is interesting.

THEOREM 5. *We can construct an infinite permutation satisfying*

$$(a_i, a_{i+1}) > \frac{1}{2}i \tag{6}$$

for all i.

On the other hand, for any permutation

$$\liminf_i \frac{(a_i, a_{i+1})}{i} \leqq \frac{61}{90}. \tag{7}$$

The right value is probably $\frac{1}{2}$, but we could not yet prove this.

Proofs

PROOF OF THEOREM 1. First we show that any permutation must contain an a_i for which

$$[a_i, a_{i+1}] \geqq (1+o(1))\frac{n^2}{4\log n}.$$

Consider the primes between $\frac{n}{2}$ and n, the number of these is about $\frac{n}{2\log n}$. Hence at least one of them has a left neighbour $\geqq (1+o(1))\frac{n}{2\log n}$, and thus the least common multiple here is $\geqq (1+o(1))\ \frac{n}{2\log n}\cdot\frac{n}{2}$.

Now we construct a permutation satisfying

$$[a_i, a_{i+1}] \leqq \{1+o(1)\}\frac{n^2}{4\log n} \tag{8}$$

for all $i \leqq n-1$.

The idea is to take the multiples of a prime p as a block, and to separate the blocks by "small" numbers. Then the l.c.m. will not be too large at the border of the