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DILATABLE OPERATOR VALUED FUNCTIONS ON C^*-ALGEBRAS

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Introduction

Our recent papers [6], [7] on moment theorems with respect to C^*-algebras offer a way to treat these questions in a setting of the dilation theory due to Halmos, Naimark and Sz.-Nagy [5]. We shall do this investigations here.

Let A be a (complex) C^*-algebra, not necessarily with unit, let G be a multiplicative semigroup in A, closed with respect to the involution of A (briefly a *-semigroup) and such that its linear span is a norm dense *-subalgebra in A. Given an operator valued function f on G, $f: G \to B(H)$, where $B(H)$ is the C^*-algebra of all bounded linear operators on the (complex) Hilbert space H, we say that f is dilatable with respect to A if there is a Hilbert space K, a continuous linear operator V of K into H and a *-representation S of A on K such that

(1) $$f(g) = VS_gV^*$$

holds for each g in G.

In the scalar valued case, when $H=\mathbf{C}$ (i.e. $\dim H=1$), V is a continuous linear functional on K, hence by the Riesz Representation Theorem there is a vector x in K such that (1) is of the form [6]

(1)′ $$f(g) = (S_gx, x) \quad (g \in G)$$

where in addition $V^*1=x$. In this case

(2)′ $$\varphi(a) = (S_ax, x) \quad (a \in A)$$

is a (unique) positive linear extension of f giving a solution of a moment theorem with respect to the C^*-algebra A.

In the previous case, similarly,

(2) $$\varphi(a) = VS_aV^* \quad (a \in A)$$

defines not only a positive linear, but also a dilatable extension of f. These two notions coincide in the case when A is commutative (Theorem 1), which is an easy consequence of [7, Theorem 4], due to the author, and generalizes a theorem of Stinespring (and Naimark too). As a corollary, we give a new characterization of subnormal operators on Hilbert space differing from that of Halmos—Bram, MacNerny and Embry (Theorem 2).

In the case when A is noncommutative, our result (Theorem 3) subsumes the previous ones and gives a common generalization of theorems of Sz.-Nagy [5] and Stinespring [8].

Dilatable functions with respect to commutative C^*-algebras

The first result is a simple consequence of our previous one proved in [7, Theorem 4].

THEOREM 1. *Let G be a multiplicative *-semigroup in a C^*-algebra A such that G generates a norm dense *-subalgebra in A. Given an operator valued function f of G with values in $B(H)$, the C^*-algebra of all bounded linear operators on the Hilbert space H, it is dilatable with respect to A if and only if there is a positive constant M such that if $0\neq x\in H$, then*

$$\text{(3)}\qquad \frac{1}{M\|x\|^2}\Big|\sum_g c_g(f(g)x,x)\Big|^2 \leqq \sum_{g,h} c_g\bar{c}_h(f(h^*g)x,x) \leqq M\|x\|^2\Big\|\sum_g c_g g\Big\|^2$$

holds for each finite sequence $\{c_g\}$ of complex numbers indexed by elements of G.

PROOF. The necessity is an easy consequence of (1) which holds by assumption:

$$\Big|\sum_g c_g(f(g)x,x)\Big|^2 = \Big|\Big(\big(\sum_g c_g S_g\big)V^*x, V^*x\Big)\Big|^2 \leqq \|V^*\|^2\|x\|^2\Big\|\big(\sum_g c_g S_g\big)V^*x\Big\|^2 =$$

$$= \|V^*\|^2\|x\|^2\sum_{g,h} c_g\bar{c}_h(VS_{h^*g}V^*x,x) = \|V^*\|^2\|x\|^2\sum_{g,h} c_g\bar{c}_h(f(h^*g)x,x) \leqq$$

$$\leqq \|V^*\|^4\|x\|^4\Big\|S\sum_g c_g g\Big\|^2 \leqq \|V^*\|^4\|x\|^4\|S\|^2\Big\|\sum_g c_g g\Big\|^2$$

where $\|S\|\leqq 1$, as S is a *-representation of the C^*-algebra A.

To prove the sufficiency of (1) assume (3) and conclude by [7, Theorem 4], (an operator valued moment theorem, if A is considered via the commutative Gelfand—Naimark Theorem as $C_0(\Omega)$, the complex valued continuous functions vanishing at infinity over the locally compact Hausdorff space Ω), that there is a positive operator valued measure $F(\cdot)$ on Ω, dilatable by our Naimark-type result [7, Theorem 2] to a spectral measure $E(\cdot)$ on a Hilbert space K which has a suitable continuous linear operator V into H such that

$$\text{(4)}\qquad F(.) = VE(.)V^*$$

holds for these two operator measures $F(\cdot)$ and $E(\cdot)$ on Ω. Define now a *-representation S of A by

$$\text{(5)}\qquad S_a = \int_\Omega a(t)E(dt)\quad (a\in A).$$

We have the desired property of S given in (1):

$$(f(g)x,x) = \int_\Omega g(t)(F(dt)x,x) = \int_\Omega g(t)(VE(dt)V^*x,x) =$$

$$= \int_\Omega g(t)E(dt)(V^*x,V^*x) = (S_gV^*x,V^*x) = (VS_gV^*x,x).$$

The proof is complete.

As a consequence we give a new characterization of subnormal operators, showing that an operator B on a Hilbert space is subnormal if and only if the