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# THE CESÀRO—DENJOY—PETTIS SCALE OF INTEGRATION

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## 1. Introduction

In an earlier paper [5] we have introduced a scale of Cesàro—Denjoy—Bochner integration. In the present paper we introduce a scale of Cesàro—Denjoy—Pettis integrals, the $C_nD_*P$ integrals, such that the strength of the integral is increased with $n$, the $C_0D_*P$ integral being the special Denjoy—Pettis integral introduced in [6].

## 2. Definition and terminologies

Throughout the paper, $R$ is the real line, $X$ is a real Banach space, $\|\cdot\|$ its norm $X^*$ its conjugate space. The definition of Peano derivative and of $AC_nG_*$ for rea function are as in [2]. We shall frequently refer to the $C_nD$ integral of [7] and to the $C_nP$ integral of [3] for real valued functions. These integrals, viz. the $C_nD$ integral and the $C_nP$ integral are equivalent (see [8]). The Lebesgue—Bochner integral and the Lebesgue—Pettis integral will be denoted by $LB$ and $LP$ respectively. Unless otherwise stated, function will mean an $X$-valued function defined on an interval $[a, b]$.

DEFINITION 2.1. Let $F\colon [a, b]\to X$ and let $\xi\in[a, b]$. Let $n$ be a positive integer. If there are constants $\alpha_1, \alpha_2, \ldots, \alpha_n\in X$ depending on $\xi$ such that

$$x^*\left[F(t)-F(\xi)-(t-\xi)\alpha_1-\ldots-\frac{(t-\xi)^n}{n!}\alpha_n\right]=o\big((t-\xi)^n\big)\quad (t\to\xi)$$

for all $x^*\in X^*$, then $\alpha_n$ is called *the weak Peano derivative of $F$ at $\xi$ of order $n$* and is denoted by $F^{\omega}_{(n)}(\xi)$. It is easily seen that if $F^{\omega}_{(n)}(\xi)$ exists then $F^{\omega}_{(k)}(\xi)$ $(1\leqq k\leqq n)$ exists. In particular $F^{\omega}_{(1)}(\xi)$ is the weak derivative of $F$ at $\xi$. For convenience we shall write $F^{\omega}_{(0)}$ to mean $F$. It is clear that if the strong Peano derivative $F_{(n)}$ (cf. [5]) exists at a point $\xi$ then $F^{\omega}_{(n)}$ also exists at $\xi$ and $F_{(n)}(\xi)=F^{\omega}_{(n)}(\xi)$.

DEFINITION 2.2. Let $F\colon [a, b]\to X$ and let $n$ be a positive integer. If there are functions $F_i\colon [a, b]\to X$; $i=1, 2, \ldots, n$ such that

$$x^*\left[F(t)-F(\xi)-(t-\xi)F_1(\xi)-\frac{(t-\xi)^2}{2!}F_2(\xi)-\ldots-\frac{(t-\xi)^n}{n!}F_n(\xi)\right]=$$
$$=o\big((t-\xi)^n\big)\quad(t\to\xi)$$

for almost all $\xi\in[a, b]$ for each $x^*\in X^*$ (the exceptional set of measure zero may vary with $x^*$), then $F_n$ is said to be *the pseudo derivative of $F$ on $[a, b]$ of order*

$n$ and is denoted by $D_P^n F$. It is easily seen that $D_P^k F\,(1 \leqq k \leqq n)$ exists if $D_P^n F$ exists and that if $F_{(k)}^\omega$ exists a.e. in $[a, b]$ then $D_P^k F$ exists in $[a, b]$ and $D_P^k F = F_{(k)}^\omega$.

DEFINITION 2.3. Let $n \geqq 0$. A function $F\colon [a, b] \to X$ is called *weakly* $AC_n G_*$ *on* $[a, b]$ if $F_{(n)}^\omega$ exists in $[a, b]$ and if the real function $x^* F$ is $AC_n G_*$ [2] for each $x^* \in X^*$.

Since $|x^* F| \leqq \|x^*\| \, \|F\|$ it is easy to verify that strong $AC_n G_*$ defined in [5] implies weak $AC_n G_*$.

## 3. Preliminary results

THEOREM 3.1. *Let* $F$ *be weakly* $AC_n G_*$ *in* $[a, b]$ *and let* $D_P^{n+1} F$ *exist in* $[a, b]$. *If* $D_P^{n+1} F = 0$ *in* $[a, b]$ *then* $F_{(n)}^\omega$ *is constant.*

PROOF. Let $x^* \in X^*$ be arbitrary. Then $x^* F$ is a real valued $AC_n G_*$ function. Since $x^* D_P^{n+1} F = 0$, $(x^* F)_{(n+1)} = 0$ a.e. So by [2; Theorem 16 coupled with Lemma 2] $(x^* F)_{(n)}$ is constant. But since $F_{(n)}^\omega$ exists in $[a, b]$, $(x^* F)_{(n)} = x^* F_{(n)}^\omega$. Hence $x^* F_{(n)}^\omega$ is constant. Let $\xi \in [a, b]$. Then $x^*\big(F_{(n)}^\omega(\xi) - F_{(n)}^\omega(a)\big) = 0$. Since $x^*$ is arbitrary, the theorem is proved.

THEOREM 3.2. *If* $F_{(n)}^\omega$, $n \geqq 1$, *exists in* $[a, b]$, *then* $F_{(k)}^\omega$ *are strongly measurable for* $k = 1, 2, \ldots, n$.

PROOF. Since $F$ is weakly continuous, it is strongly measurable (cf. [6] or [4' p. 73]). Since $F_{(1)}^\omega$ exists in $[a, b]$, for each $t \in [a, b]$ and each $x^* \in X^*$

$$\lim_{h \to 0} x^* \frac{1}{h} [F(t+h) - F(t)] = x^* F_{(1)}^\omega(t).$$

Taking any sequence $\{h_r\}$ which converges to 0 we get a sequence of strongly measurable functions $\left\{\frac{1}{h_r} [F(t+h_r) - F(t)]\right\}$ which converges to $F_{(1)}^\omega(t)$ weakly everywhere. So, by ([4; p. 74, Theorem 3.5.4], $F_{(1)}^\omega$ is strongly measurable. Thus the theorem is true for $n = 1$. Suppose that it is true for $n = m-1$. Then since $F_{(k)}^\omega$ is strongly measurable for $k = 0, 1, \ldots, m-1$ and since by the existence of $F_{(m)}^\omega$ we have

$$\lim_{h \to 0} x^* \left[\frac{m!}{h^m} \left\{F(t+h) - F(t) - h F_{(1)}^\omega(t) - \ldots - \frac{h^{m-1}}{(m-1)!} F_{(m-1)}^\omega(t)\right\}\right] = x^* F_{(m)}^\omega(t)$$

for each $t \in [a, b]$ and each $x^*$, applying similar argument as above, $F_{(m)}^\omega$ is strongly measurable. The proof is thus complete by induction.

DEFINITION 3.3. A function $f\colon [a, b] \to X$ is said to be $C_n D_* P$ *(Cesàro—Denjoy—Pettis) integrable* if there is a weakly $AC_n G_*$ function $F\colon [a, b] \to X$ such that $D_P^{n+1} F$ exists in $[a, b]$ and $D_P^{n+1} F = f$ on $[a, b]$. Then $F_{(n)}^\omega(t)$ is called an indefinite $C_n D_* P$ integral of $f$ and $F_{(n)}^\omega(b) - F_{(n)}^\omega(a)$ is its definite $C_n D_* P$ integral in $[a, b]$ and is denoted by

$$(C_n D_* P) \int_a^b f(t)\, dt.$$