Acta Math. Hung.
46 (3—4) (1985), 285—295.

GENERAL LINEAR SUMMATION OF THE VILENKIN—FOURIER SERIES

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Some special summation of Vilenkin—Fourier series [7] of functions in $L^p(0, 1)$ $(1 \leqq p \leqq \infty)$ were investigated by S. Yano, A. Efimov, I. Yastrebova, S. Blumin, Su Weiyi and the author etc. (see [1]—[5]). In [1] Blumin also discussed the general summation. This paper is devoted to extending some results in [5] for the case $1 \leqq p \leqq \infty$ and some specific discussions for the case $1 < p < \infty$.

The notions and the symbols in this paper are basically the same as in [5], but some of them have been revised.

§ 1. Notions and symbols

1. Let $\mathbf{N} := \{1, 2, \ldots\}$, $\mathbf{P} := \{0, 1, 2, \ldots\}$, $\{m_j\}_{j \in \mathbf{P}}$ be a sequence of integers each greater than 1, $\sup m_j < \infty$. $\mathbf{Z}_j := \{0, 1, \ldots, m_j - 1\}$, $M_0 := 1$, $M_{j+1} := m_j M_j$ $(j \in \mathbf{P})$.

2. If $x, y \in [0, 1)$ and their expansions respectively are $x = \sum_{j=0}^{\infty} x_j M_{j+1}^{-1}$, $y = \sum_{j=0}^{\infty} y_j M_{j+1}^{-1}$ $(x_j, y_j \in \mathbf{Z}_j)$, then let $x \ominus y := \sum_{j=0}^{\infty} (x_j - y_j) \pmod{m_j} M_{j+1}^{-1}$.

3. Let $\varphi_k(x) = \exp \frac{2\pi i}{m_k} x_k$, $\psi_k(x) := \prod_{j=0}^{\infty} (\varphi_j(x))^{k_j}$, where $x = \sum_{j=0}^{\infty} x_j M_{j+1}^{-1}$ $(x_j \in \mathbf{Z}_j)$, $k = \sum_{j=0}^{\infty} k_j M_j \in \mathbf{P}$ $(k_j \in \mathbf{Z}_j)$. $\{\psi_k\}_{k \in \mathbf{P}}$ is called the Vilenkin function system.

4. $D_n(x) := \sum_{j=0}^{n-1} \psi_j(x)$, $F_n(x) := \frac{1}{n} \sum_{j=1}^{n} D_j(x)$ $(n \in \mathbf{N})$.

5. Let $f, g \in L^p[0, 1)$ $(1 \leqq p \leqq \infty)$, $f\hat{}(k) := \int_0^1 f(t) \bar{\psi}_k(t)\, dt$ $(k \in \mathbf{N})$, $(f * g)(x) := \int_0^1 f(x \ominus t) g(t)\, dt$, $S_n(f) := f * D_n$ $(n \in \mathbf{N})$.

6. $\omega(f, \delta) := \omega(L^p[0, 1), f, \delta) := \sup_{0 \leqq h < \delta} \|f(\cdot \ominus h) - f(\cdot)\|_p$ $(f \in L^p[0, 1),\ 1 \leqq p \leqq \infty,\ \delta > 0)$.

7. Let G and H be functions or functionals defined on a set U. If there exist positive numbers α_1 and α_2 such that for all $x \in U$, $\alpha_1 G(x) \leqq H(x) \leqq \alpha_2 G(x)$, we will denote it by $G \sim H$.

8. Let $f \in L^p[0, 1)$ $(1 \leqq p \leqq \infty)$, $T_r^{\langle \alpha \rangle}(t) := \sum_{k=0}^{M_r - 1} k^\alpha \psi_k(t)$ (α is real, $0^\alpha := 1$ if $\alpha < 0$).

If there exists $g \in L^p[0,1)$ such that $\lim_{r\to\infty} \|T_r^{\langle\alpha\rangle} * f - g\|_p = 0$, then if $\alpha > 0$, g is called the (strong) derivative of order α of f in $L^p[0,1)$; if $\alpha < 0$, g is called the (strong) integral of order $(-\alpha)$ of f in $L^p[0,1)$. In both cases g will be denoted by $T^{\langle\alpha\rangle} f$.

§ 2. The case $1 \leqq p \leqq \infty$

Blumin [1] gave the following result:

If $f \in L^p[0,1)$ $(1 \leqq p \leqq \infty)$, $K_n = 1 + \sum_{k=1}^{n-1} c_k \psi_k$ $(n \in \mathbf{N})$, then

$$\|f - f * K_n\|_p \leqq c\left\{(1 + \|K_n\|_1) E_n(f) + \sum_{l=0}^{r} \Big\| \sum_{k=1}^{M_{l+1}-1} (1-c_k)\psi_k \Big\|_1 E_{M_l}(f)\right\} \tag{2.1}$$

where $M_r \leqq n < M_{r+1}$, $E_n(f) := \inf\{\|f-g\|_p : g\hat{\ }(k) = 0\ (\mathbf{N} \ni k \geqq n)\}$ and $C > 0$ is an absolute constant.

We will give another estimate by a simpler method. First we show

LEMMA 1. *If $f \in L^p[0,1)$ $(1 \leqq p \leqq \infty)$, $g \in L^1[0,1)$, $g\hat{\ }(k) = 0$ $(k = 0, 1, 2, \ldots, M_r - 1,\ r \in \mathbf{N})$, then*

$$\|f * g\|_p \leqq \omega\left(f, \frac{1}{M_r}\right) \|g\|_1. \tag{2.2}$$

PROOF. Since for any $k \in \mathbf{P}$,

$$f\hat{\ }(k)\, g\hat{\ }(k) = \big(f\hat{\ }(k) - (S_{M_r}(f))\hat{\ }(k)\big) g\hat{\ }(k),$$

by the uniqueness theorem of Fourier transform, we get $f * g = (f - S_{M_r}(f)) * g$. Hence by [8]

$$\|f * g\|_p \leqq \|f - S_{M_r}(f)\|_p \|g\|_1 \leqq \omega\left(f, \frac{1}{M_r}\right) \|g\|_1.$$

THEOREM 1. *If $f \in L^p[0,1)$ $(1 \leqq p \leqq \infty)$, $K_\varrho \in L^1[0,1)$ (ϱ belongs to a set of indices), $K_\varrho\hat{\ }(0) = 1$, $K_\varrho\hat{\ }(k) := C_k(\varrho) := C_k$ $(k \in \mathbf{N})$, then for any positive integer r*

$$\|f - f * K_\varrho\|_p \leqq \omega\left(f, \frac{1}{M_r}\right) + \sum_{l=0}^{r-1} \Big\| \sum_{k=M_l}^{M_{l+1}-1} (1-c_k)\psi_k \Big\|_1 \omega\left(f, \frac{1}{M_l}\right) + \tag{2.3}$$

$$+ \sum_{l=r}^{\infty} \Big\| \sum_{k=M_l}^{M_{l+1}-1} c_k \psi_k \Big\|_1 \omega\left(f, \frac{1}{M_l}\right).$$

PROOF. Since the sequence of the M_r-th partial sums of a Vilenkin—Fourier series of an integrable function is convergent in $L^1[0,1)$ (see e.g. [7]), i.e. $K_\varrho = 1 + \sum_{l=0}^{\infty} \sum_{k=M_l}^{M_{l+1}-1} c_k \psi_k$, we have for all $s \geqq r$ $(s \in \mathbf{N})$

$$\|f - f * K_\varrho\|_p \leqq \|f - S_{M_r}(f)\|_p + \Big\| f * \sum_{l=0}^{r-1} \sum_{k=M_l}^{M_{l+1}-1} (1-c_k)\psi_k \Big\|_p +$$

$$+ \Big\| f * \sum_{l=r}^{s} \sum_{k=M_l}^{M_{l+1}-1} c_k \psi_k \Big\|_p + \Big\| f * \sum_{l=s+1}^{\infty} \sum_{k=M_l}^{M_{l+1}-1} c_k \psi_k \Big\|_p.$$