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STRONG APPROXIMATIONS OF RENEWAL PROCESSES AND THEIR APPLICATIONS

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1. Introduction

Let (X, Y), $\{(X_n, Y_n),\ Y_n=(Y_n^{(1)}, \ldots, Y_n^{(d)}),\ n\geqq 1\}$ be a sequence of random vectors with values in R^{d+1}. Many authors (see, for example, the Introduction and Chapter 2 in Csörgő and Révész [8]) studied the rate of strong approximation of the partial sums $(U(t), S(t))$,

$$U(t)=\sum_{i=1}^{[t]} X_i, \quad S(t)=\sum_{i=1}^{[t]} Y_i$$

by a $(d+1)$-dimensional Wiener process. Horváth [13] obtained that a strong invariance principle for the partial sums of independent, indentically distributed random variables (i.i.d.r.v's) with positive expectation always implies a strong approximation for the corresponding renewal process. The renewal process, being the inverse of the partial sum process is defined as

$$N(t)=\inf\{s\colon U(s)>t\},$$

$$=\infty, \quad \text{if} \quad \{s\colon U(s)>t\}=\emptyset.$$

First we show that Theorem 2.1 in [13] remains true if we drop the independence and identical distribution assumption on the summands. A joint approximation of $N(t)$ and $S(N(t))$ will be proved in Section 2. Partial sums indexed by a renewal process appear in the mathematical theory of risk processes and queuing processes. Gut and Janson [10] gave some other interesting examples of the use of $S(N(t))$ in the theory of chromatography, classical renewal theory, chemistry, physics, replacement policies and economics.

In the last section we consider some applications of our main theorems. We obtain that our method gives the best possible joint approximation of $U(t)$, $S(t)$, $N(t)$ and $S(N(t))$.

We can assume without loss of generality that our probability space $(\Omega, \mathcal{A}, P)$ is so rich that every r.v. and all processes introduced later on are defined on it. Throughout this paper we use the maximum norm in R^k denoted by $\|x\|_k=\max_{1\leqq i\leqq k}|x_i|$, $x=(x_1, \ldots, x_k)$. The transpose of a row-vector x is a column-vector denoted by x^T. Let $a\wedge b=\min(a, b)$, $a\vee b=\max(a, b)$. We use the abbrevations $\xi_T\overset{\text{a.s.}}{=}o(a(T))$ and $\xi_T\overset{\text{a.s.}}{=}O(b(T))$, where $\{\xi_T, a(T), b(T), T\geqq 0\}$ are stochastic processes, to mean that

$$\lim_{T\to\infty}\xi_T/a(T)=0 \quad \text{a.s.}$$

and

$$P\{\limsup_{T\to\infty} |\xi_T|/b(T) = \infty\} = 0,$$

respectively. We say that $a(T)$ is not greater than $b(T)$ almost surely $\big(a(T) \overset{\text{a.s.}}{\leqq} b(T)\big)$, if for almost all $\omega\in\Omega$ there is an integer $n_0=n_0(\omega)$ such that $a(T)\leqq b(T)$ for $T\geqq n_0$.

2. Strong approximations of the renewal process and the partial sums indexed by the renewal process

Several authors proved strong invariance principles for sums of random variables or random vectors under different conditions. We do not want to summarize these results in a single statement and hence we are not going to list the different sets of conditions (moment and dependence conditions) allowing such strong approximations. We will simply assume that the partial sums can be approximated by a Gaussian process and strong invariance principles for $N(t)$ and $S(N(t))$ will follow from this assumption of strong approximation.

CONDITION A. *We can define a* $(d+1)$*-dimensional Wiener process*

$$\{W(t) = (W^{(1)}(t), \ldots, W^{(d+1)}(t)),\ t\geqq 0\},\quad EW(t) = 0,\ EW^T(t)W(s) = \Gamma \min(t, s)$$

such that

$$\sup_{0\leqq t\leqq T} \big\|\big(U(t)-\mu t,\ S(t)-mt\big)-W(t)\big\|_{d+1} \overset{\text{a.s.}}{=} o\big(r(T)\big), \tag{2.1}$$

where $\Gamma=\{\gamma_{i,j}\}$, $1\leqq i,j\leqq d+1$ *is a nonsingular covariance matrix,* (μ, m) *is a constant vector,* $r(T)$ *is nondecreasing, regularly varying at infinity and*

$$r(T) = O\big((T\log\log T)^{1/2}\big). \tag{2.2}$$

For the sake of simplicity we use the notation $\sigma^2=\gamma_{1,1}$. Condition A in the following theorem (and Condition B in Theorem 2.2 below) is meant only for the first component.

THEOREM 2.1. *If* $\mu>0$ *then Condition* A *implies that*

$$\sup_{0\leqq t\leqq T} |\mu^{-1}t-N(t)-\mu^{-1}W^{(1)}(\mu^{-1}t)| \overset{\text{a.s.}}{=} o\big(r(T)\big),$$

if

$$(T\log\log T)^{1/4}(\log T)^{1/2} = o\big(r(T)\big) \tag{2.3}$$

and

$$\limsup_{T\to\infty}(T\log\log T)^{-1/4}(\log T)^{-1/2} \sup_{0\leqq t\leqq T} |\mu^{-1}t-N(t)-\mu^{-1}W^{(1)}(\mu^{-1}t)| =$$

$$= 2^{1/4}\sigma^{3/2}\mu^{-7/4}\quad a.s.,$$

if

$$r(T) = O\big((T\log\log T)^{1/4}(\log T)^{1/2}\big). \tag{2.4}$$

It is very important that the partial sums and the renewal process are approximated by the *same* Wiener process. It follows from this theorem that the rate of