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A MOMENT THEOREM FOR CONTRACTIONS ON HILBERT SPACES

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Given a subset X of a Hilbert space H which spans the space H, and a function f: $\mathbf{Z}\times X\to H$, where $\mathbf{Z}$, as usual, stands for the set of integers, one can ask whether there exists a contraction T on the Hilbert space H such that

$$T_n x = f(n, x) \quad (n\in\mathbf{Z}, x\in X) \tag{1}$$

holds, where T_n is defined for $n\in\mathbf{Z}$ as follows:

$$T_n = \begin{cases} T^n & \text{if} \quad n \geqq 0, \\ T^{*|n|} & \text{if} \quad n < 0. \end{cases} \tag{1'}$$

For a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on the Hilbert space H, with an extension $T_{-t}=T_t^*$ $(t\geqq 0)$ to $\mathbf{R}$, the corresponding problem is that given a function f: $\mathbf{R}\times X\to H$, under what condition does there exist a continuous semigroup $\{T_t\}$ of contractions such that

$$T_t x = f(t, x) \quad (t\in\mathbf{R}, x\in X). \tag{2}$$

The present note gives an answer to these problems. It is in connection with the preceding papers [1], [2] on this subject. [1] deals with equation (1) required for $X=\{x_0\}$ and $n\geqq 0$ only, i.e., the equation $T^n x_0 = x_n$ $(n=1, 2, \ldots)$, with given $\{x_n\}_0^\infty \subset H$; and analogously for the continuous one parameter case T_t $(t\geqq 0)$. On the other hand, [2] considers the case when the operators T_n we are seeking for are not derived from some contraction T as in (1') but rather from some unitary operator U on a Hilbert space H' and from an operator V: $H'\to H$ of the form $T_n = VU^nV^*$ $(n\in\mathbf{Z})$; and analogously, $T_t = VU_tV^*$ $(t\in\mathbf{R})$ where U_t is a continuous one parameter group of unitaries. (In fact, [2] treats even more general groups and $*$-semigroups.)

For unitary dilation theory we refer to Sz.-Nagy [3].

THEOREM 1. *There exists a contraction T satisfying* (1) *if and only if the function f satisfies*

$$f(0, x) = x \quad (x\in X), \tag{3}$$

$$\big(f(n, x), f(m, y)\big) = \big(f(n-m, x), y\big) \quad (m, n\in\mathbf{Z}\colon mn \leqq 0;\ x,y\in X), \tag{4}$$

$$\Big\|\sum_{n,x} c_{n,x} f(n, x)\Big\|^2 \leqq \sum_{m,y}\sum_{n,x} \bar{c}_{m,y} c_{n,x} \big(f(n-m, x), y\big) \tag{5}$$

where $\{c_{n,x}\}$ $(n\in\mathbf{Z}, x\in X)$ is an arbitrary finite double sequence of complex numbers.

THEOREM 2. *There exists a continuous family* $\{T_t\}$ *of contractions satisfying* (2) *if and only if f is continuous in its first variable and satisfies the identities* (3), (4), (5) *with* m, n *taking their values in* $\mathbf{R}$.

PROOF OF NECESSITY. 1. Let T be a contraction on the Hilbert space H and let U be a unitary dilation on a Hilbert space K containing H as a subspace. Then for the orthogonal projection P of K onto H we have

$$T_n x = PU^n x \quad (n\in\mathbf{Z}, x\in H) \tag{6}$$

where T_n is defined by (1'). Let further $\{C_{n,x}\}$ be a finite double sequence of complex numbers indexed by elements of the set $\mathbf{Z}\times X$. Then we have by (1) and (6), for any x, y in X,

$$f(0, x) = T_0 x = x,$$

$$(f(n,x), f(m,y)) = (T^n x, T^{*|m|}y) = (T^{n+|m|}x, y) = (T^{n-m}x, y) =$$
$$= (f(n-m, x), y) \quad \text{if} \quad m<0,\ n\geqq 0,$$

$$(f(n,x), f(m,y)) = (T^{*|n|}x, T^m y) = (T^{*(|n|+m)}x, y) =$$
$$= (T^{*|n-m|}x, y) = (f(n-m, x), y) \quad \text{if} \quad m\geqq 0,\ n<0;$$

and

$$\Big\|\sum_{n,x} c_{n,x} f(n,x)\Big\|^2 = \Big\|\sum_{n,x} c_{n,x} T_n x\Big\|^2 = \Big\|P\sum_{n,x} c_{n,x} U^n x\Big\|^2 \leqq \Big\|\sum_{n,x} c_{n,x} U^n x\Big\|^2 =$$
$$= \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(U^n x, U^m y) = \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(U^{n-m}x, y) =$$
$$= \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(U^{n-m}x, Py) = \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(PU^{n-m}x, y) =$$
$$= \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(T_{n-m}x, y) = \sum_{m,y}\sum_{n,x} \bar c_{m,y} c_{n,x}(f(n-m, x), y).$$

2. The case of a continuous semigroup $\{T_t\}$ of contractions can be dealt with in a similar way by using the corresponding minimal dilation $\{U_t\}$.

PROOF OF SUFFICIENCY. 1. Let F_0 be the (complex) linear space of all finite double sequences $\{c_{n,x}\}$ $(n\in\mathbf{Z}, x\in X)$ of complex numbers with the shift operation

$$U_0\{c_{n,x}\} := \{c'_{n,x}\}, \quad \text{where} \quad c'_{n,x} = c_{n-1,x} \quad (n\in\mathbf{Z},\ x\in X).$$

Introduce a semi-inner product $\langle.\,,.\rangle$ in F_0 by

$$\langle\{c_{n,x}\}, \{d_{m,y}\}\rangle := \sum_{m,y}\sum_{n,x} \bar d_{m,y} c_{n,x}(f(n-m, x), y). \tag{7}$$

Positive semi-definiteness follows from (5). It also follows from (5) that the linear map V_0 defined by

$$V_0\{c_{n,x}\} := \sum_{n,x} c_{n,x} f(n,x) \quad (\{c_{n,x}\}\in F_0)$$

is a contraction from F_0 into the Hilbert space H.

Denote F the Hilbert space resulting from F_0 by factoring with respect to the null space of $\langle.\,,.\rangle$ and then by completing with respect to the norm inherited. At the same time U_0 induces a unitary operator U on the Hilbert space F and V_0 induces a