Acta Math. Hung.
49 (3—4) (1987), 353—363.

AFFINELY EMBEDDABLE CONVEX SETS

T. BISZTRICZKY and J. SCHAER (Calgary)

A subset A of the real projective plane P^2 s defined to be convex in P^2, if there is a line L in P^2 disjoint from A and A is a convex subset of the affine plane $P^2 \setminus L$; cf. [2] and [4].

Let $\mathscr{A}_r$ be a finite collection of mutually disjoint convex sets in P^2. In [3], N. H. Kuiper determines conditions under which there exists a line in P^2 meeting every element of $\mathscr{A}_r$. Presently, we determine when there exists a line in P^2 not meeting any element of $\mathscr{A}_r$, or equivalently, when each element of $\mathscr{A}_r$ is convex (and thus contained) in the same affine restriction of P^2.

This problem, interestingly enough, is related to Helly's theorem, and for pointing out this connection and how it is obtained, we wish to express our thanks to Dr. I. Bárány.

The connection is based upon the extension of polarity from R^3 to P^2, Since P^2 is obtained from S^2 by identifying antipodal points, a convex set $A \subset P^2$ is obtained by identifying disjoint connected sets $S^+(A)$ and $S^-(A)$ in S^2 with $S^+(A) = -S^-(A)$. As there is a convex cone $C(A) \subset R^3$ with $C(A) \cap S^2 = S^+(A)$ and the polar of a convex cone is again a convex cone, the polar of $C(A)$ determines a set $B \subset P^2$ which we call the polar of A. The set B is semi-convex, that is, any two points of B can be joined by a line segment contained in B; cf. [2]. If B is disjoint from a line of P^2 then of course, B is convex.

Let $\mathscr{B}_r = \{B_1, B_2, \ldots, B_r\}$ be the family of polars of $\mathscr{A}_r = \{A_1, A_2, \ldots, A_r\}$. Then by polarity, $\bigcap_{l=1}^{r} B_l \neq \emptyset$ if and only if there is a line in P^2 not meeting any $A_i \in \mathscr{A}_r$.

Now our main result for $\mathscr{A}_r$ restated for $\mathscr{B}_r$ is as follows.

If for any three B_i, B_j and B_k in $\mathscr{B}_r$, $r \geqq 4$, no connected (convex) component of $B_i \cap B_j$ is contained in B_k then $\bigcap_{l=1}^{r} B_l \neq \emptyset$.

Preliminaries

We denote the points and lines of P^2 by $p, q, \ldots$ and $L, M, \ldots$ respectively, and the line spanned by distinct points p and q by $\langle p, q \rangle$.

If A is a subset of P^2 and L is a line in P^2 disjoint from A then A is contained in the affine restriction $P^2 \setminus L$, and we denote by $H_L(A)$ the *convex hull* of A in $P^2 \setminus L$. As stated, A is then *convex* (in P^2) if $H_L(A) = A$. If A is convex and disjoint from the lines M and N then $H_M(A) = A = H_N(A)$.

If $\mathscr{A}_r=\{A_1, A_2, \ldots, A_r\}$ is a collection of $r\geqq 2$ convex sets in P^2, we say that $\mathscr{A}_r$ is *affinely embeddable* if there is a line L in P^2 such that $L\cap\bigcup_{i=1}^{r} A_i=\emptyset$; that is, if each $A_i\to\mathscr{A}_r$ is convex in the same affine restriction $P^2\setminus L$.

We wish to determine necessary and sufficient conditions for $\mathscr{A}_r$ to be affinely embeddable. There is a necessary condition independent of r but the sufficient conditions which we determine depend on whether $r=2$, $r=3$ or $r\geqq 4$.

1. Lemma. *Let $\mathscr{A}_r$ be a collection of r convex sets of P^2; $r\geqq 2$. If $\mathscr{A}_r$ is affinely embeddable then the intersection of any two elements of $\mathscr{A}_r$ is connected.*

Proof. Let $\{A_j, A_k\}\subset\mathscr{A}_r$ such that $A_j\cap A_k$ is disconnected. Since both sets are convex, $A_j\cap A_k$ consists of two components. Choose points p and q, one from each component. Then

$$\langle p, q\rangle = (\langle p, q\rangle\cap A_j)\cup(\langle p, q\rangle\cap A_k)$$

and every line in P^2 meets $A_j\cup A_k$; thus $\mathscr{A}_r$ is not affinely embeddable. □

We note that the necessary condition in Lemma 1 is not sufficient. In fact, even if the elements of $\mathscr{A}_r$ are mutually disjoint, $\mathscr{A}_r$ need not be affinely embeddable as the following trivial example shows: Let p and q be distinct points; then they are the end points of two disjoint open segments of $\langle p, q\rangle$, say B_1 and B_2. Then $A_1=B_1$ and $A_2=\bar{B}_2$ are convex in P^2, $A_1\cap A_2=\emptyset$, $A_1\cup A_2=\langle p, q\rangle$ and $\{A_1, A_2\}$ is not affinely embeddable. Next we note the following trivial sufficient condition for $\mathscr{A}_r$ to be affinely embeddable.

2. Lemma. *Let $\mathscr{A}_r$ be a collection of $r\geqq 2$ closed convex sets of P^2 such that there is an A_i and an A_j in $\mathscr{A}_r$ with the property that one of their common supporting lines meets neither $A_i\cap A_j$ nor any other element $\mathscr{A}_r$. Then $\mathscr{A}_r$ is affinely embeddable.*

3. Lemma. *Let A_1 and A_2 be closed convex sets in P^2 such that $A=A_1\cap A_2$ is connected. Then $\{A_1, A_2\}$ is affinely embeddable.*

Proof. If $A=\emptyset$ then Lemma 3 follows from a proof by Kuiper in [3]. Hence let $A\neq\emptyset$. Since A_i is convex, there is a line M_i such that $M_i\cap A_i=\emptyset$; $i=1, 2$. We may assume that $M_1\neq M_2$, the sets A_1, A_2 and A are distinct, $M_1\cap\operatorname{int} A_2\neq\emptyset$ and $M_2\cap\operatorname{int} A_1\neq\emptyset$. Let $M_1\cap M_2=\{q\}$ and denote by Q and Q' the closed half-planes of P^2 bounded by M_1 and M_2; since $A\cap(M_2\cup M_2)=\emptyset$, $A\subset\operatorname{int} Q'$ say. Let $L\subset Q'$, $M_1\neq L\neq M_2$ and set $A_i^*=Q\cap A_i$; $i=1, 2$.

As $A\subset\operatorname{int} Q'$ and $M_1\cap\operatorname{int} A_2\neq\emptyset\neq M_2\cap\operatorname{int} A_1$, A_1^* and A_2^* are non-empty disjoint closed convex sets in $P^2\setminus L$. Hence (cf. [1]) there exist distinct lines $\tilde{N}_1$ and $\tilde{N}_2$ in $P^2\setminus L$ such that

i) each is a common supporting line of A_1^* and A_2^* and

ii) each separates A_1^* and A_2^* in $P^2\setminus L$.

Setting $\tilde{M}_i=M_i\setminus L$ $(i=1, 2)$, our construction can be easily followed in the affine plane $P^2\setminus L$: $\tilde{M}_1$ and $\tilde{M}_2$ are parallel lines, $Q\setminus L$ is the parallel strip bounded by $\tilde{M}_1$ and $\tilde{M}_2$ and $A_1^*\cup A_2^*\subset Q\setminus L$.

Let $N_i\subset P^2$ such that $N_i\setminus L=\tilde{N}_i$ $(i=1, 2)$ and let $N_1\cap N_2=\{p\}$. Let P and P' denote the closed half-planes of P^2 determined by N_1 and N_2. By (ii), $\tilde{N}_1\cap\tilde{N}_2=$