Acta Math. Hung.
55 (1—2) (1990), 111—123.

ON THE SUMMATION OF SOME EXPANSIONS

I. JOÓ* (Budapest)

The aim of the present paper is to prove some convergence theorems for eigenfunction expansions connected with differential operators, on the basis of the latest results [1, 2] of V. Komornik and the author.

We need the following lemma, which is interesting in itself.

LEMMA 1. *Let* (μ_n), (ν_n) *be two sequences of real numbers. Suppose* $\mu_n\to\infty$ $(n\to\infty)$.

a) *If* $\mu_n-\nu_n=O(1)$ *then for every* $f\in L^1(0,2\pi)$

$$S_{\mu_n}(f,x)-S_{\nu_n}(f,x)=o(1)\quad (n\to\infty)$$

uniformly on $[0,2\pi]$.

b) *If* $\mu_n-\nu_n\neq O(1)$, *then there exists* $f\in L^1(0,2\pi)$ *such that*

$$\sup_{n\geqq 1}|S_{\mu_n}(f,x)-S_{\nu_n}(f,x)|=\infty\quad (x\in[0,2\pi]).$$

Here $S_r(f,x):=(D_r*f)(x)$, *and*

$$D_r(t):=\frac{\sin\left(r+\frac{1}{2}\right)t}{2\pi\sin\frac{t}{2}},\quad 0\leqq t\leqq 2\pi.$$

PROOF. a) We have

$$S_{\mu_n}(f,x)-S_{\nu_n}(f,x)=\frac{1}{\pi}\int_0^{2\pi}f(t)\frac{\sin\frac{\mu_n-\nu_n}{2}(t-x)}{2\sin\frac{t-x}{2}}\cos\frac{\mu_n+\nu_n+1}{2}(t-x)\,dt.$$

Define for $f\in L^1(0,2\pi)$ the set

$$\mathscr{F}_f:=\{f_{x,n}:\ n\in\mathrm{N},\ 0\leqq x\leqq 2\pi\},$$

$$f_{x,n}(t):=f(t)\frac{\sin\frac{\mu_n-\nu_n}{2}(t-x)}{2\sin\frac{t-x}{2}}.$$

* Work supported by the Hungarian National Science Foundation for Research, grant No. 1157.

This set is precompact in $L^1(0, 2\pi)$. Indeed, we can define for any $\varepsilon > 0$ an ε-lattice in $\mathcal{F}_f$ as follows. Let

$$\frac{2k\pi}{n} \leqq x \leqq \frac{2(k+1)\pi}{n} \quad \text{and} \quad \frac{l}{n} \leqq \frac{\mu_n - \nu_n}{2} \leqq \frac{l+1}{n},$$

then

$$\left\| f_{x,n}(t) - f(t) \frac{\sin \frac{l}{n}\left(t - \frac{2k\pi}{n}\right)}{\sin \frac{t - \frac{2k\pi}{n}}{2}} \right\|_{L^1} < \varepsilon \quad \text{if} \quad n > n_0(\varepsilon).$$

So $\mathcal{F}_f$ is precompact. Introduce the functionals $F_{n,x}$: $L^1(0, 2\pi) \to \mathbf{C}$,

$$F_{n,x}(g) := \frac{1}{\pi} \int_0^{2\pi} g(t) \cos \frac{\mu_n + \nu_n + 1}{2} (t - x)\, dt.$$

Since $\frac{\mu_n + \nu_n + 1}{2} \to \infty$, we have by the Riemann—Lebesgue lemma that $F_{n,x}(g) \to 0$ $(n \to \infty)$ for any fixed $g \in L^1(0, 2\pi)$ and $x \in [0, 2\pi]$. We show that this convergence is uniform in $x \in [0, 2\pi]$ and $g \in \mathcal{F}_f$. Suppose the contrary, i.e. that $|F_{n_k, x_k}(g_k)| \geqq \delta > 0$ for some $n_k \in \mathbf{N}$, $x_k \in [0, 2\pi]$, $g_k \in \mathcal{F}_f$. Taking subsequences we can obtain by the precompactness of $\mathcal{F}_f$, that $g_k \to g_0 \in L^1(0, 2\pi)$, $x_k \to x_0 \in [0, 2\pi]$. Obviously $\|F_{n,x}\| \leqq 1$, hence $|F_{n_k, x_k}(g_0)| \geqq \frac{\delta}{2}$ $(k \geqq k_0)$. Since $F_{n_k, x_0}(g_0) \to 0$ $(k \to \infty)$, hence for $k \geqq k_0$ we have

$$\frac{\delta}{4} \leqq |F_{n_k, x_k}(g_0) - F_{n_k, x_0}(g_0)| =$$

$$= \left| \frac{1}{\pi} \int_0^{2\pi} g_0(t) \left\{ \cos \frac{\mu_{n_k} + \nu_{n_k} + 1}{2} (t - x_k) - \cos \frac{\mu_{n_k} + \nu_{n_k} + 1}{2} (t - x_0) \right\} dt \right| \leqq$$

$$\leqq \left| \frac{1}{\pi} \int_0^{x_k - x_0} g_0(t) \cos \frac{\mu_{n_k} + \nu_{n_k} + 1}{2} (t - x_0)\, dt \right| +$$

$$+ \left| \frac{1}{\pi} \int_{2\pi - (x_k - x_0)}^{2\pi} g_0(t) \cos \frac{\mu_{n_k} + \nu_{n_k} + 1}{2} (t - x_k)\, dt \right| +$$

$$+ \left| \frac{1}{\pi} \int_{x_k - x_0}^{2\pi} \{g_0(t - (x_k - x_0)) - g_0(t)\} \cos \frac{\mu_{n_k} + \nu_{n_k} + 1}{2} (t - x_0)\, dt \right| = I_1 + I_2 + I_3,$$