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A HAJÓS—KELLER TYPE RESULT ON FACTORIZATION OF FINITE CYCLIC GROUPS

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1. Introduction. Let G be a finite abelian group written additively and let $B, A_1, \ldots, A_n$ be its subsets. If every $b \in B$ is uniquely expressible in the form

$$b = a_1 + \ldots + a_n, \quad a_1 \in A_1, \ldots, a_n \in A_n,$$

then we will express this fact such that the equation

$$B = A_1 + \ldots + A_n$$

is a factorization of B. If all A_i contain the zero element, then we will speak of normed factorization. The subset

$$\{0, g, 2g, \ldots, (q-1)g\}$$

will be denoted briefly by $[g, q]$ and will be called a simplex provided the positive integer q is not greater than the order of g. Here q is the length, g is the generator and qg is the terminating element of the simplex. We denote the order of g by $|g|$, and the generatum of the subset A of G, that is the smallest subgroup which contains A, will be denoted by $\langle A \rangle$.

The most well-known result on the field of factorization of finite abelian groups is the so-called Hajós' theorem which asserts that in a simplex factorization of a finite abelian group one of the simplices must be a subgroup.

In this theorem the lengths of the simplices may be reduced to primes so the following result of L. Rédei can be viewed as a generalisation for it. In a normed factorization of a finite abelian group by its subsets of prime cardinalities one of the factors is a subgroup.

The algebraic form of the so-called Keller's conjecture proposes a different generalisation of Hajós' theorem. Namely, if

$$G = H + [g_1, q_1] + \ldots + [g_n, q_n] \tag{1}$$

is a normed factorization of a finite abelian group G, then

$$(H-H) \cap \{q_1 g_1, \ldots, q_n g_n\} \neq \emptyset,$$

where $H-H = \{h'-h \colon h', h \in H\}$.

The geometrical background and the history of these problems can be found in [4].

The purpose of this paper is to investigate the following question. From the factorization (1) does it follows that

$$q_j g_j \in \langle H \rangle \quad \text{for some } j, \quad 1 \leqq j \leqq n?$$

Obviously, this follows from Keller's conjecture and this concludes Hajós' theorem, i.e. the case $H=0$.

Although Keller's conjecture is undecided even in the case of cyclic groups, in the remaining part of the paper we will prove the proposed problem for cyclic groups.

2. Result. We need a result on the replaceable factors of factorizations.

LEMMA ([2] Proposition 3 p. 370). *If*

$$G = [g, q] + A$$

is a factorization of a finite abelian group G and q is a prime, then the simplex $[g, q]$ *can be replaced by* $[rg, q]$ *whenever r is prime to q.*

Now we are ready to prove the main result of this paper.

THEOREM. *If*

$$G = H + [g_1, q_1] + \ldots + [g_n, q_n] \tag{2}$$

is a normed factorization of the finite cyclic group G, then there exists a j, $1 \leqq j \leqq n$ *such that* $q_j g_j \in \langle H \rangle$.

PROOF. Let $K = \langle H \rangle$. If $K = G$, then we are done. Thus we may suppose that $K \neq G$. Next note that

$$[g, uv] = [g, u] + [ug, v]$$

is a factorization and the terminating element of the left hand side simplex is the same as the terminating element of the second simplex on the right hand side. This shows that we may replace the simplices by simplices of prime lengths that is we may restrict our investigation to the case of q_i's being primes.

Let $p_1^{a_1} \ldots p_t^{a_t}$ be the prime factorization of $|G|$. According to the fundamental theorem of finite abelian groups, G is the direct sum of cyclic groups of orders $p_1^{a_1}, \ldots, p_t^{a_t}$, respectively. Let $C_1, \ldots, C_t$ be these groups and let $x_1, \ldots, x_t$ be the corresponding basis elements.

Thus the factorization derived from (2) is

$$G = H + [g_{11}, p_1] + \ldots + [g_{1r_1}, p_1] + \ldots + [g_{t1}, p_t] + \ldots + [g_{tr_t}, p_t]. \tag{3}$$

In virtue of the Lemma the simplex $[g_{ij}, p_i]$ can be replaced by $[c_{ij}, p_i]$, where $c_{ij} \in C_i$. Consequently c_{ij} is a multiple of x_i, say $c_{ij} = a_{ij} x_i$. Moreover we may suppose that a_{ij} is a power of p_i and further that $p_i c_{ij} \neq 0$. Thus the elements c_{ij} are taken from the elements

$$\begin{matrix} x_1, & p_1 x_1, & \ldots, & p_1^{a_1-2} x_1 \\ \vdots & \vdots & & \vdots \\ x_t, & p_t x_t, & \ldots, & p_t^{a_t-2} x_t. \end{matrix}$$