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ON THE DIVERGENCE OF SOME FUNCTION SERIES

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This paper is devoted to the study of the divergence of Fourier series. In the four sections below we deal with the a.e., resp. the norm divergence of $S_{\mu_n}f-S_{\nu_n}f$, where $\mu_n-\nu_n\neq O(1)$; the a.e. divergence for signed Toeplitz summations and another norm divergence problem. We also formulate two corresponding problems.

1. Investigate first the pointwise divergence of a sequence of type $S_{\mu_n}f-S_{\nu_n}f$.

LEMMA 1. *Suppose $\{\mu_n\}$ and $\{\nu_n\}$ are natural numbers such that $\mu_n\to+\infty$, $\nu_n\to\to+\infty$ $(n\to\infty)$ and define*

$$(1)\qquad T_n:=S_{\mu_n}-S_{\nu_n};\quad S_N=\sum_{|n|\leqq N}c_ne^{inx};\quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx.$$

Suppose that there exists $f_1\in L_1(0,2\pi)$ such that

$$\limsup_{n\to\infty}|T_nf_1(x)|>0$$

on a set of positive measure. Then there exists $\delta>0$ and for any $M>0$ there exists a polynomial $f=f_M$ satisfying $\|f\|_1\leqq K$ with a constant K independent of M and δ; further

$$(2)\qquad \sup_n|T_nf(x)|>M\quad(x\in E)$$

where $E\subset[0,2\pi]$ is a set of Lebesgue measure $|E|>\delta$.

PROOF. We can suppose that

$$\limsup_{n\to\infty}|T_nf_1(x)|\geqq c_0>0\quad(x\in E_1)$$

for a set E_1, $|E_1|>0$. Fix an arbitrary $M^*>0$. Let[1]

$$f(x):=M^*\cdot\vartheta_N(f_1-\sigma_m(f_1))(x)$$

where N and m will be specified later and ϑ_N denotes the N-th de la Vallée-Poussin means. If m is large enough, $m>m_0(M^*,f_1)$, then

$$\|f\|_1\leqq cM^*\|f_1-\sigma_m(f_1)\|_1\leqq K.$$

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[1] $\vartheta_N:=\dfrac{S_{N+1}+\ldots+S_{2N}}{N}$, $\quad\sigma_N:=\dfrac{S_0+\ldots+S_N}{N+1}$.

Suppose now that m, M^* are fixed and vary the number N. Clearly

$$n > N_1(m) \Rightarrow \nu_n, \mu_n > m.$$

We know that

$$\limsup_{\substack{n\to\infty \\ n>N_1}} |T_n f_1(x)| \geqq c_0 > 0 \quad (x \in E_1)$$

so there exists a set E, $|E| > \frac{|E_1|}{2}$ and a number N_2 such that

$$\sup_{N_1<n<N_2} |T_n f_1(x)| \geqq c_0/2 \quad (x \in E).$$

Here N_2 depends only on f_1, μ_n, ν_n. Let N be so large that $N > \mu_n, \nu_n$ if $N_1 < n < N_2$. Then

$$T_n f = M^*\{[S_{\mu_n} f_1 - \sigma_m f_1] - [S_{\nu_n} f_1 - \sigma_m f_1]\} = M^* T_n f_1,$$

consequently

$$\sup_{N_1<n<N_2} |T_n f(x)| = M^* \sup |T_n f_1(x)| \geqq \frac{c_0}{2} M^* \quad (x \in E)$$

and then

$$M := \frac{c_0}{2} M^*, \quad \delta := \frac{|E_1|}{2}$$

satisfies the statement of Lemma 1.

LEMMA 2. *Suppose that* $\mu_n, \nu_n \to +\infty$ *are natural numbers and* $|\mu_n - \nu_n| \to +\infty$. *Then there exists* $\delta > 0$ *and for any* $M > 0$ *there exists a polynomial* g *with* $\|g\|_1 \leqq 3\pi$ *and such that*

$$\sup_n |T_n g(x)| > M$$

in a set of x *of measure* $\geqq \delta$.

PROOF. We can suppose $\mu_n > \nu_n$. Fix $n \in \mathbf{N}$ and define

$$a_i := \frac{4\pi i}{2n+1} \quad (i = 0, 1, \ldots, n)$$

and

$$g(x) := \frac{1}{n} \sum_{\substack{i=1 \\ (i\in I)}}^{n}{}' V_{\mu_{k_i}}(x - a_i)$$

where

$$V_\mu(x) = \frac{1}{2} + \sum_{j=1}^{\mu} \cos jx + \sum_{j=\mu+1}^{2\mu} \left(1 - \frac{j-1}{\mu+1}\right) \cos jx$$

is the de la Vallée-Poussin kernel and Σ' means that the summation is restricted to the indices $i \in I$, where the set I will be given later. Define a sequence k_i with the properties

$$n^4 < \nu_{k_0}, \quad (\mu_{k_i} >) \nu_{k_i} > 2\mu_{k_{i-1}} \quad (i = 1, 2, \ldots).$$