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CHARACTERIZATION OF SUBPROJECTION SUBOPERATORS

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Introduction

After P. R. Halmos [2], a suboperator is a bounded linear transformation from a subspace of a Hilbert space into the whole space. A couple of problems initiated also by Halmos in the paper just mentioned arises when one asks for a characterization of subpositive, subprojection e.t.c. suboperators, that is for ones there are positive, projection e.t.c. operators that extend these suboperators. Of course, subselfadjoint suboperators are, in view of the now classical theorem of M. G. Krein, symmetric suboperators. A simple proof of this fact (together with extension not increasing the norm as usual) can be found in Z. Sebestyén [3]. Here an independent characterization of subpositive suboperators is proved as a starting point for the selfadjoint case. This turned out to be the natural approach.

As a matter of fact the so called Schwarz inequality is proved to be characteristic for subpositive suboperators in the author's paper [3]. In the present note we show that the Schwarz identity (with constant one) characterizes precisely the subprojection suboperators (Theorem 1).

As a corollary we get the characterization of Halmos [2, Proposition 3] and in a remark we prove the same result of Halmos for subpositive suboperators [2, Corollary 2] using our result.

Factorizations through projection are proved in Corollaries 2 and 3.

Characterization of subprojections

Given a (complex) Hilbert space H, a (closed) subspace H_0 in it and a suboperator $Q: H_0 \to H$, we are interested in searching for a (selfadjoint) projection P on H which restricted to H_0 is Q itself.

THEOREM 1. *Let $Q: H_0 \to H$ be a suboperator. Q is a subprojection if and only if the identity*

$$\|Qx\|^2 = (Qx, x) \quad (x \in H_0) \tag{1}$$

holds true.

PROOF. An operator P on a Hilbert space H is an orthogonal projection if and only if it is selfadjoint and idempotent:

$$P^* = P = P^2. \tag{2}$$

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In other words
$$\|Px\|^2 = (Px, Px) = (P^2x, x) = (Px, x)$$
holds true for any x in H. In the case when Q is the restriction of P to H_0 this reduces to (1).

On the other hand if we assume (1) to hold for Q, the approach of [3], [4] applies: define a semi-inner product $\langle , \rangle$ on H_0 by

$$\langle x, y\rangle := (Qx, y) \quad (x, y \in H_0). \tag{3}$$

Then another Hilbert space K arises by taking completion of the quotient space H_0/N with respect to the norm inherited from the inner product (denoted by the same symbol) on this space, where N is the nullspace of $\langle , \rangle$ in H_0. For x in H_0, $(x+N)$ is the corresponding vector in H_0/N so that

$$V(x+N) := Qx \quad (x \in H_0) \tag{4}$$

defines on the dense subset H_0/N of K a map $V\colon H_0/N \to H$ which is an isometry as well. Indeed,
$$\|V(x+N)\|^2 \overset{(4)}{=} \|Qx\|^2 \overset{(1)}{=} (Qx, x) \overset{(3)}{=} \langle x+N, x+N\rangle$$
holds true for all x in H_0. Here we use step by step (4), (1) and (3) respectively. We have thus a unique isometry, denoted also by V, of K into H as an extension of the former V. The desired projection of H will be $P := VV^*$. First, this is selfadjoint (moreover positive) and idempotent since V^*V is the identity operator on K by the isometry of V so that
$$P^2 = V(V^*V)V^* = VV^* = P.$$
That P restricted to H_0 is Q is a consequence of the characteristic identity

$$V^*x = x+N \quad (x \in H_0), \tag{5}$$

we have discovered in our previous works. It is implied by the identity (for any y in H_0):
$$\langle y+N, V^*x\rangle = \big(V(y+N), x\big) \overset{(4)}{=} (Qy, x) = \langle y+N, x+N\rangle \quad (y \in H_0).$$
Indeed (5) implies (using (4))
$$Px = V(V^*x) \overset{(5)}{=} V(x+N) \overset{(4)}{=} Qx$$
as desired. The proof is complete.

COROLLARY 1 (Halmos). *$Q\colon H_0 \to H$ is a subprojection if and only if $A^* = A$ and $A - A^2 = B^*B$ hold true for the operators $A\colon H_0 \to H_0$, $B\colon H_0 \to H \ominus H_0$ that represent Q as a "column matrix"* $\begin{pmatrix} A \\ B \end{pmatrix}$.

PROOF. In the representation just mentioned

$$Qx = Ax \oplus Bx, \quad Ax \in H_0, \quad Bx \in H \ominus H_0 \tag{6}$$