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A GENERAL SADDLE POINT THEOREM AND ITS APPLICATIONS

Z. SEBESTYÉN (Budapest)

Let X and Y be nonempty sets, f and g be real-valued functions on the Cartesian product $X\times Y$ of these sets. A point (x, y) in $X\times Y$ is said to be a *saddle point* of the functions f, g if

(SP) $$g(u, y) \leqq f(x, v) \quad \text{for every} \quad (u, v) \quad \text{in} \quad X\times Y$$

holds true. For a single function f the well-known notion of saddle point follows here by letting $g\equiv f$ in (SP). It should also be noted that the existence of a saddle point implies the following minimax inequality

(MMI) $$\inf_{y\in Y}\sup_{x\in X} g(x, y) \leqq \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

In the case when $f\leqq g$, especially when g equals f, the latter property is known as the statement of the two variable generalized version of the celebrated von Neumann's minimax theorem, namely

(MME) $$\inf_{y\in Y}\sup_{x\in X} g(x, y) = \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

Our aim is to prove a general but rather elementary theorem first on the existence of saddle points (Theorem 1), secondly, as a consequence, on the existence of minimax inequality and equality respectively — giving necessary and sufficient conditions for them. Our condition is general enough and not only of convexity type. The results so obtained are a common generalization of our previous ones and many other known theorems of concave-convex type. Our approach is essentially the same as our earlier one. We use the finite dimensional separation argument for disjoint convex sets in a similar but essentially simpler way as in [1, Theorem 2.5.1] and Riesz's well-known theorem concerning a common point of compact sets with finite intersection property. The compactness here follows by Alexander's subbase theorem [6].

Concerning minimax type inequalities see S. Simmons [10], J. Kindler [5] and Z. Sebestyen [8, 9]. Minimax theorems are e.g. in Belakrishnan [1], Z. Sebestyen [7, 8, 9], I. Joó [3] and I. Joó—L. L. Stachó [4].

Let now f, g be two real-valued functions defined on the Cartesian product $X\times Y$ of two nonempty sets X, Y. As a notation, for a nonemty set $K\subset X\times Y$, for a point (u, v) in $X\times Y$ and for a positive real number c let

$$K^c_{u,v} = \{(x, y)\in K\colon\ 0 \leqq f(x, v)-g(u, y)+c\}.$$

This is why for a point (x, y) in $X\times Y$ to be a saddle point is nothing else but each $K^c_{u,v}$ being nonempty for the one point set $K=\{(x, y)\}$.

THEOREM 1. *Let f, g be real-valued functions on $X\times Y$. There exists a saddle point for f, g if and only if there exists a nonempty set $K\subset Y\times Y$ such that :*

(1) $$\min_{(u,v)\in G} \sum_{(x,y)\in F} \lambda(x,y)[f(x,v)-g(u,y)] \leqq \sup_{(x,y)\in K} \min_{(u,v)\in G} [f(x,v)-g(u,y)]$$

for all finite sets $F\subset K$, $G\subset X\times Y$ and a probability measure λ on F;

(2) $$0 \leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in K} [f(x,v)-g(u,v)] \leqq \sup_{(x,y)\in K} \sum_{(u,v)\in G} \mu(u,v)[f(x,v)-g(u,y)]$$

for every finite set $G\subset X\times Y$ and a probability measure μ on G;

(3) *if $D\subset(0,+\infty)\times X\times Y$ has the property that for any (x,y) in K there exists (c,u,v) in D with $f(x,v)-g(u,y)+c<0$, then a finite subset of D exists with the same property.*

PROOF. Assume first that a point (x,y) in $X\times Y$ is a saddle point for the functions f, g on $X\times Y$. The one point subset $K=\{(x,y)\}$ of $X\times Y$ clearly satisfies conditions (1), (2) and (3)

To prove the sufficiently let K be as in the assumption. Let further $U^c_{u,v}=K\setminus K^c_{u,v}$ be the complements in K of the subsets $K^c_{u,v}$ introduced before.

Topologize K by taking $\{U^c_{u,v}: (c,u,v)\in(0,+\infty)\times X\times Y\}$ as a family of open subbase for this topology. Condition (3) says that if K is covered by a subfamily $\{U^c_{u,v}: (c,u,v)\in D\}$ then K is also covered by a finite subcollection of the family indexed by D. By Alexander's well-known subbase lemma K is thus compact in the topology so introduced. But the subsets $K^c_{u,v}$ of K are thus closed hence compact with respect to this topology on K. Now a point (x,y) in $X\times Y$ satisfies (SP) if and only if

$$0 \leqq f(x,v)-g(u,y)+c \quad \text{holds for all} \quad (c,u,v)\in(0,+\infty)\times X\times Y,$$

in other words (x,y) belongs to each of $K^c_{u,v}$. To prove that a saddle point exists is therefore nothing else but to prove that the sets $K^c_{u,v}$ have a common point. But the compactness of $K^c_{u,v}$'s allows us, refering to Riesz, to prove the finite intersection property of the family $K^c_{u,v}$. Let $0<c_i$, $(u_i,v_i)\in X\times Y$ for $i=1,2,\dots,n$ have a finite family of subsets $K^{c_i}_{u_i,v_i}$ in K indexed by $i=1,2,\dots,n$. Since with $c=\{\min c_i: 1\leqq i\leqq n\}$

$$K^c_{u_i,v_i} \subset K^{c_i}_{u_i,v_i} \quad \text{for} \quad i=1,2,\dots,n,$$

$\bigcap_{i=1}^n K^c_{u_i,v_i}\neq\emptyset$ will imply the desired nonvoid intersection property for the chosen finite family $\{K^{c_i}_{u_i,v_i}: i=1,2,\dots n\}$. Assume the contrary: $\bigcap_{i=1}^n K^c_{u_i,v_i}=\emptyset$. Then we conclude that for any (x,y) in K there exists a natural number i, $1\leqq i\leqq n$ such that $(x,y)\notin K^c_{u_i,v_i}$, i.e. $f(x,v_i)-g(u_i,y)+c<0$.

This implies the following property:

(4) $$\min_{1\leqq i\leqq n} [f(x,V_i)-g(u_i,y)] < -c \quad \text{for any} \quad (x,y) \quad \text{in} \quad K.$$

Let now Φ_c be the R^n-valued function on K defined as follows:

$$\Phi_c(x,y) := \big(f(x,v_1)-g(u_1,y)+c, \dots, f(x,v_n)-g(u_1,y)+c\big).$$