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ON THE MEANS OF THE ARGUMENT OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

L. GROENEWALD (Bloemfontein)

1. Let $\zeta(s)$ denote the Riemann zeta–function and put

$$\pi S(t) = \Delta_L \arg \zeta(s)$$

where Δ_L denotes the variation in the argument of $\zeta(s)$ along the polygonal line L extending from 2 to $2 + it$ and then to $\frac{1}{2} + it$. Since $\arg\zeta(2) = 0$, we can express $S(t)$ in the form $\pi S(t) = \arg\zeta\left(\frac{1}{2} + it\right)$ provided the argument is defined by continuous variation along L ([1], p. 98).

In [2] Ghosh proved for $k = 1$ and k an even number that

$$(1) \qquad \int\limits_{T}^{T+H} |S(t)|^k \, dt \sim \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right)\left(\frac{1}{2\pi}\right)^k H(\log\log T)^{k/2}, \quad T \to \infty$$

with an error term which holds uniformly in $k \ll (\log\log T)^{1/6}$.

Ghosh's main theorem in [2] on sign changes of $S(t)$ in the interval $(T, T + H)$ is deduced from these latter estimates. For recent conditional results on sign changes of $S(t)$, see [3].

Ghosh [2] mentions without proof that the asymptotic relation (1) can be extended to all integral values of k. It is the aim of this paper to prove Ghosh's claim.

THEOREM. *Let H be a function of T such that $T^\alpha \leqq H(T) \leqq T$, where $\frac{1}{2} < \alpha \leqq 1$ for all $T \geqq 1$. Then, for any positive integer k*

$$\int\limits_{T}^{T+H} |S(t)|^k \, dt \sim \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right)\left(\frac{1}{2\pi}\right)^k H(\log\log T)^{k/2}, \quad T \to \infty.$$

2. We shall need the following:

LEMMA 2.1.

$$\int\limits_{0}^{\infty} \frac{1}{u^2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(2u)^{2j}(2k+2j)!}{(2j)!\,(k+j)!} \, du = 2^{2k+1} k! \sqrt{\pi}, \quad k = 0, 1, 2, \ldots.$$

PROOF. Since

$$\frac{(2k+2j)!}{(2j)!(k+j)!}=$$
$$=\frac{(2k+2j)(2k+2j-2)\dots(2j+2)(2j)!(2k+2j-1)(2k+2j-3)\dots(2j+1)}{(k+j)(k+j-1)\dots(j+1)(2j)!j!}=$$
$$=\frac{2^k(2k+2j-1)(2k+2j-3)\dots(2j+1)}{j!}$$

if $k \geqq 1$, it follows, on substituting z for $2u$, that the integral above can be written as

$$2^{k+1}\int_0^\infty \frac{1}{z^2}F_{2k-1}(z)\,dz$$

where

(2) $$F_{-1}(z)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^{2j}}{j!}=1-e^{-z^2}$$

and

$$F_{2k-1}(z)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^{2j}}{j!}(2k+2j-1)(2k+2j-3)\dots(2j+1),\quad k\geqq 1.$$

Note that

(3) $$F_{2k+1}(z)=\frac{1}{z^{2k}}\frac{d}{dz}\left(z^{2k+1}F_{2k-1}(z)\right)\quad\text{if}\quad k\geqq 0.$$

Every $F_{2k-1}(z)$ can be written in the form

(4) $$F_{2k-1}(z)=\sum_{i=0}^{k}a_{ki}z^iF_{-1}^{(i)}(z)$$

where the a_{ki} are constants. Indeed, (4) is obvious if $k=0$. If (4) holds for $k=n$, then

$$F_{2n+1}(z)=\frac{1}{z^{2n}}\frac{d}{dz}\left(z^{2n+1}\sum_{i=0}^{n}a_{ni}z^iF_{-1}^{(i)}(z)\right)=$$
$$=\sum_{i=0}^{n}(2n+1+i)a_{ni}z^iF_{-1}^{(i)}(z)+\sum_{i=0}^{n}a_{ni}z^{i+1}F_{-1}^{(i+1)}(z)$$