Acta Math. Hung.
59 (1–2) (1992), 25–32.

ON THE CONTROL OF A RECTANGULAR MEMBRANE

I. JOÓ (Budapest)

1. Let $\Omega = (0,a)\times(0,b)$ be a rectangle and consider the following control problem:

$$u_{tt} = \Delta u + \sum_{j=1}^{N} \delta((x,y) - P_j)v_j, \quad (x,y) \in \Omega \tag{1}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega\times(0,T)} = \sum_{j=N+1}^{M} \delta(s - S_j)v_j, \quad s \in \partial\Omega. \tag{2}$$

Here $u(t,x,y)$ gives the height of the point $(x,y) \in \Omega$ in time t, $P_1,\dots,P_N \in \Omega$, $S_{N+1},\dots,S_M \in \partial\Omega$ and

$$v_j(t) \in L^2(0,T) \quad (j = 1,\dots,M)$$

are the controls. We suppose further that the initial state is relaxed, i.e.

$$u(0,x,y) = u_t(0,x,y) = 0 \quad \text{on} \quad \Omega. \tag{3}$$

Define further the reachability set

$$R(T) := \{(u(T,.,.), u_t(T,.,.)) : v_j \in L^2(0,T); j = 1,\dots,M\}$$

and consider the following system of eigenfunctions:

$$-\Delta\varphi_{mn} = \lambda_{mn}\varphi_{mn}, \quad \frac{\partial\varphi_{mn}}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega.$$

It has solutions

$$\varphi_{mn}(x,y) = \gamma_{mn}\cos m\frac{\pi}{a}x\cos n\frac{\pi}{b}y, \quad \lambda_{mn} = \left(m\frac{\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)^2, \quad (m,n = 0,1,2,\dots),$$

forming a complete orthonormal system in $L^2(\Omega)$ if

$$\gamma_{mn} := \begin{cases} \dfrac{2}{\sqrt{ab}} & \text{if } m,n \neq 0 \\ \dfrac{1}{\sqrt{ab}} & \text{if } m = n = 0 \\ \sqrt{\dfrac{2}{ab}} & \text{if one of } m,n \text{ is } 0. \end{cases}$$

Introduce the spaces

$$W_r := \left\{ f = \sum_{m,n} c_{mn}\varphi_{mn} : \| f \|^2_{W_r} := |c_{00}|^2 + \sum_{(m,n)\neq(0,0)} |c_{mn}|^2 \lambda^r_{mn} < \infty \right\},$$

$$\mathcal{H}_r := W_{r+1} \oplus W_r.$$

First we prove

THEOREM 1. *For any control the solution of* (1)–(3) *satisfies* $(u, u_t) \in \in C([0,T], \mathcal{H}_r)$ *for* $r < -3/4$.

We recall that the system (1)–(3) is approximately controllable in time T if for all $r < -3/4$ $R(T) \subset \mathcal{H}_r$ is dense in $\mathcal{H}_r$. The system is approximately controllable in nonbounded finite time if $\bigcup_{T<\infty} R(T)$ is dense in $\mathcal{H}_r$ for all $r < -3/4$.

THEOREM 2. (a) *The system* (1)–(3) *is not approximately controllable in any time* $T < \infty$ *(i.e.* $R(T) \cap \mathcal{H}_r$ *is not dense in* $\mathcal{H}_r$ *for any* $r \in \mathbf{R}$*).*

(b) *The system* (1)–(3) *is approximately controllable in nonbounded finite time i.e.* $\bigcup_{T<\infty} R(T)$ *is dense in* $\mathcal{H}_r$ *if and only if all* λ_{mn} *are different (i.e.* a^2/b^2 *is not rational) and if all* e_{mn} *are nonzero.*

As a corollary we get that, contrary to the onedimensional case of vibrating strings ([1], [2]) the reachability set $R(T)$ does not give up growing for large T.

We give to (1)–(3) the following interpretation. The function $u(t,x,y)$ satisfies (1)–(3) if for every function $z(t,x,y) \in C^2$ with

$$z(T,.,.) = z_t(T,.,.) = 0, \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega\times(0,T)} = 0,$$

the following relation holds:

$$\int_0^T \int_\Omega u(z_{tt} - \Delta z) = \int_0^T \left(\sum_{j=1}^N z(., P_j)v_j + \sum_{j=N+1}^M z(., S_j)v_j \right). \tag{4}$$

We ask for the solution u in the form

$$u(t,x,y) = \sum_{m,n\in\mathbf{N}} c_{mn}(t)\varphi_{mn}(x,y);$$

then

$$c_{mn} = \int_\Omega u\varphi_{mn}.$$