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THE NUMERICAL SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

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1. Introduction

In this paper we are going to approximate the solution of the nonlinear differential equation

$$(1)\qquad \begin{cases} y^{(m)}(x) = f\left[x, y(x), y'(x), \dots, y^{(m-1)}(x)\right], & x \in [0;1], \\ y^{(j)}(0) = y_0^{(j)} \ (j = 0, 1, \dots, m-1;\ m \geqq 2) \end{cases}$$

supposing

$$(1.1)\qquad f\left[x, y(x), y'(x), \dots, y^{(m-1)}(x)\right] \in C^{(r)}([0;1])$$

where r is a fixed integer,

$$(1.2)\qquad \left| f^{(q)}\left[x, y_1, y_1', y_1'', \dots, y_1^{(m-1)}\right] - f^{(q)}\left[x, y_2, y_2', y_2'', \dots, y_2^{(m-1)}\right] \right| \leqq$$
$$\leqq L \cdot \sum_{i=0}^{m-1} \left| y_1^{(i)} - y_2^{(i)} \right| \quad (q = 0, 1, \dots, r)$$

(Lipschitz condition), where

$$f^{(0)} = f,\ f^{(q+1)} = f_x^{(q)} + f_y^{(q)} \cdot y' + f_{y'}^{(q)} \cdot y'' + \dots + f_{y^{(m-2)}}^{(q)} \cdot y^{(m-1)} + f_{y^{(m-1)}}^{(q)} \cdot f$$
$$(q = 0, 1, 2, \dots, r-1).$$

The problem of approximating the solution of nonlinear differential equations has been always of special interest. By spline functions the Cauchy problem $y' = f(x, y)$ was discussed by F. R. Loscalzo and T. D. Talbot [1], [2] and Th. Fawzy [3]. Some problems of a second order differential equation was solved by Gh. Micula [4], [5] by Th. Fawzy [6], [7] and by J. Győrvári [8], [9], [10].

The main point of our method is that approximated values of $y^{(q)}(x_k)$ are constructed by the help of $y_0^{(j)}$ $(j = 0, 1, \dots, m-1)$ and the function f, then using these approximate values $\overline{y}_k^{(q)}$ $(k = 0, 1, 2, \dots, n;\ q = 0, 1, 2)$ the solution of (1) and its derivatives are approximated up to the $(m+r)$–th order by the spline function of type $(0;1;2)$ defined in [6]. In the approximation theorems we use the average moduli defined in [11].

2. The first approximation process

2.1. Definition of the approximate values $\overline{y}_k^{(q)}$. Let

$$x_k = \frac{k}{n}; \quad h = \frac{1}{n}; \quad x_{k+1/2} = x_k + \frac{h}{2} \quad (k = 0, 1, \dots, n);$$

$$\omega(f^{(r)}, x, h) = \sup_{t_1, t_2 \in \left[x - \frac{h}{2}; x + \frac{h}{2}\right] \cap [0,1]} \left| f^{(r)}(t_1) - f^{(r)}(t_2) \right|,$$

$$\tau\left(f^{(r)}, h\right) = \int_0^1 \omega\left(f^{(r)}, x, h\right) dx.$$

DEFINITION.

$$\overline{y}_0^{(j)} := y_0^{(j)} \quad (j = 0, 1, \dots, m-1),$$
$$\overline{y}_0^{(m+q)} := f^{(q)}\left[x_0, y_0, y_0', \dots, y_0^{(m-1)}\right] \quad (q = 0, 1, \dots, r),$$
$$\overline{y}_k^{(q)} := G_k^{(q)}(x_k) \quad (k = 1, 2, \dots, n-1;\ q = 0, 1, \dots, m+r),$$
$$\overline{y}_n^{(q)} := G_{n-1}^{(q)}(x_n) \quad (q = 0, 1, \dots, m+r),$$

where

(2.1.1)

$$G_0(x) := \sum_{j=0}^{m-1} \frac{y_0^{(j)}}{j!}(x - x_0)^j + \sum_{q=0}^{r} \frac{f^{(q)}\left[x_0, y_0, y_0', \dots, y_0^{(m-1)}\right]}{(m+q)!}(x - x_0)^{m+q}$$

if $x_0 \leqq x \leqq x_1$;

$$G_k(x) := \sum_{j=0}^{m-1} \frac{G_{k-1}^{(j)}(x_k)}{j!}(x - x_k)^j + \tag{2.1.2}$$

$$+ \sum_{q=0}^{r} \frac{f^{(q)}\left[x_k, G_{k-1}(x_k), G_{k-1}'(x_k), \dots, G_{k-1}^{(m-1)}(x_k)\right]}{(m+q)!}(x - x_k)^{m+q}$$

if $x_k \leqq x \leqq x_{k+1}$ $(k = 1, 2, \dots, n-1)$.

One can see that

$$G_{k-1}^{(j)}(x_k) = G_k^{(j)}(x_k) \quad (k = 1, 2, \dots, n-1;\ j = 0, 1, \dots, m-1).$$