Acta Math. Hung.
62 (3–4) (1993), 277–283.

ON INTEGRAL FORMULAS FOR CONVEX DOMAINS

W. CIEŚLAK (Lublin), S. KOSHI (Sapporo) and J. ZAJĄC (Łódź)

1. Generalized Holditch's theorem

Let C be a closed convex regular C^1-curve in the Euclidean plane, parametrized by arc length s. We denote by L the perimeter of C. Moreover, we denote by $\langle\ ,\ \rangle$ and $[\ ,\ ]$ the Euclidean scalar product and the determinant, repsectively.

Let $\nu\colon [0,+\infty) \to R$ be a function satisfying the condition

$$(1)\qquad \begin{cases} \nu \text{ is differentiable and } \nu' > 0, \\ s < \nu(s) < s + L \text{ for all } s \geqq 0, \\ \nu(s+L) = \nu(s) + L \text{ for all } s \geqq 0. \end{cases}$$

With a curve C, $s \to z(s) = (x(s), y(s))$ for all $0 \leqq s \leqq L$ and a function ν we associate the vector field q defined as follows:

$$(2)\qquad q(s) = z(s) - z(\nu(s)) \quad \text{for } 0 \leqq s \leqq L.$$

We denote by $\alpha(s)$ the angle contained between the vectors $z_0 = z(0)$ and $q(s)$. Differentiating the relation

$$\cos\alpha = \frac{\langle q, z_0\rangle}{|q||z_0|}$$

we obtain

$$-\alpha' \sin\alpha = \frac{1}{|q|^3|z_0|}\{\langle q', z_0\rangle |q|^2 - \langle q, z_0\rangle\langle q, q'\rangle\} =$$

$$= \frac{1}{|q|^3|z_0|}[q, z_0][q, q'] =$$

$$= \frac{1}{|q|^3|z_0|}(-|q||z_0|\sin\alpha)[q, q'] = \frac{-\sin\alpha}{|q|^2}[q, q']$$

and

$$(3)\qquad \alpha' = \frac{[q, q']}{|q|^2}.$$

Hence we immediately get the following integral formula:

$$\oint \frac{[q,q']}{|q|^2} ds = 2\pi. \tag{4}$$

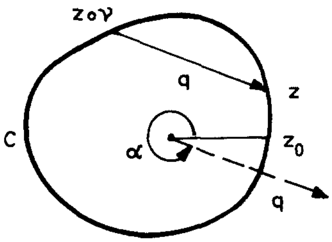

Fig. 1

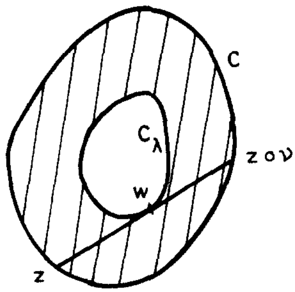

Fig. 2

Now, we give some application of the above formula.

Let us fix an arbitrary number $\lambda \in (0,1)$. We consider the curve C_λ, $s \to w(s) = (1-\lambda)z(s) + \lambda z(\nu(s))$ for $0 \leqq s \leqq L$. We note that C_λ is a closed curve. We find the area A of the region bounded by C and C_λ. By S and S_λ we denote the areas of the regions bounded by C and C_λ respectively. Using the Green formula we obtain

$$2A = 2S - 2S_\lambda = \oint([z,z'] - [w,w'])ds =$$

$$= \oint([z,z'] - (1-\lambda)^2)[z,z'] - \lambda(1-\lambda)[z, z'\circ\nu]\nu' -$$

$$-\lambda(1-\lambda)[z\circ\nu, z'] - \lambda^2[z\circ\nu, z'\circ\nu]\nu')ds =$$

$$= \oint((2\lambda-\lambda^2)[z,z'] - \lambda^2[z\circ\nu, z'\circ\nu]\nu')ds -$$

$$-\lambda(1-\lambda)\oint([z\circ\nu, z'] + [z, z'\circ\nu]\nu')ds =$$

$$= (2\lambda - 2\lambda^2)2S - \lambda(1-\lambda)\oint([z\circ\nu - z, z'] + [z,z'] -$$

$$-[z\circ\nu - z, z'\circ\nu]\nu' + [z\circ\nu, z'\circ\nu]\nu')ds =$$

$$= 4\lambda(1-\lambda)S - \lambda(1-\lambda)4S - \lambda(1-\lambda)\oint(-[q,z'] + [q, z'\circ\nu]\nu')ds =$$

$$= \lambda(1-\lambda)\oint[q, z\circ\nu\cdot\nu' - z']ds = \lambda(1-\lambda)\oint[q,q']ds.$$