ON THE SUMS OF POWERS OF COMPLEX NUMBERS

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1. Let $z_1, \dots, z_k$ be complex numbers and put

$$s_\nu = z_1^\nu + \cdots + z_k^\nu. \tag{1.1}$$

In § 2 of this note I shall prove that if

$$\max_{1 \leqq j \leqq k} |z_j| \geqq 1, \tag{1.2}$$

then

$$\max_{1 \leqq \nu \leqq 2k-1} |s_\nu| \geqq 1. \tag{1.3}$$

This result is the best possible of its kind since (i) if $z_1 = 1, z_2 = \cdots = z_k = 0$, then $|s_\nu| = 1$ for all ν, and (ii) for each k M. BOWEN has given sets of z_j such that $|s_\nu| < 1$ for $1 \leqq \nu \leqq 2k-2$ (see TURÁN's book[1]). (Such sets will also be given readily by our analysis.)

In the context of the problem (see [1]) it is natural to consider the more general sums

$$\sigma_\nu = b_1 z_1^\nu + \cdots + b_k z_k^\nu$$

where $b_1, \dots, b_k$ are positive numbers not depending on ν. It turns out that the situation is here unexpectedly different, as we shall show in §§ 3, 4. Finally, in § 5 we shall discuss other applications of our methods.

For the relation of our results to DIRICHLET's theorem and other topics we refer to [1].

2. Suppose, first, that $z_1, \dots, z_k$ are the roots of an equation

$$z^k + a_1 z^{k-1} + \cdots + a_k = 0 \tag{2.1}$$

with real $a_1, \dots, a_k$. Then the s_ν are real and satisfy the Newton identities

$$\left.\begin{aligned} s_1 &= -a_1, \\ s_2 + a_1 s_1 &= -2a_2, \\ &\cdots\cdots\cdots \\ s_k + a_1 s_{k-1} + \cdots + a_k s_1 &= -k a_k, \\ s_{k+1} + a_1 s_k + \cdots + a_k s_2 &= 0. \end{aligned}\right\} \tag{2.2}$$

[1] P. TURÁN, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953), Akadémiai Kiadó.

It follows at once that

(2. 3) $$\max_{1 \leqq \nu \leqq k+1} s_\nu \geqq 0,$$

since otherwise (2.2) would give in order $a_1 > 0, a_2 > 0, \ldots, a_k > 0$ and then the last eqration of (2. 2) would give a contradiction.

Now let $z_1, \ldots, z_k$ be any complex numbers. Then

$$2\,\Re\, s_\nu = z_1^\nu + \cdots + z_k^\nu + \bar{z}_1^\nu + \cdots + \bar{z}_k^\nu$$

and the argument above shows that

(2. 4) $$\max_{1 \leqq \nu \leqq 2k+1} \Re\, s_\nu \geqq 0.$$

We may now prove the result enunciated in § 1 that (1. 2) implies (1. 3). We may suppose without loss of generality that $|z_1| = \max |z_j|$ and so, by considering z_j/z_1 instead of z_j, that

$$1 = z_1 = \max |z_j|.$$

Then, by (2. 4) with $k-1$ for k, we have

(2. 5) $$\max_{1 \leqq \nu \leqq 2k-1} |s_\nu| \geqq \max_{1 \leqq \nu \leqq 2k-1} \Re\, s_\nu = 1 + \max_{1 \leqq \nu \leqq 2k-1} \Re (z_2^\nu + \cdots + z_k^\nu) \geqq 1.$$

It remains to show that (2. 4), (2. 5) are the best possible. If

$$z_j = \exp\left(\frac{2\pi i j}{2k+1}\right) \qquad (i^2 = -1),$$

we have

$$\Re\, s_\nu = -\frac{1}{2} \qquad (1 \leqq \nu \leqq 2k).$$

Now put

$$z_1 = 1, \quad z_j = \varepsilon \exp\left(\frac{2\pi i (j-1)}{2k-1}\right) \qquad (j > 1)$$

where $\varepsilon > 0$ is small. Then

$$\Re\, s_\nu = 1 - \frac{1}{2}\varepsilon^\nu, \quad |\Im\, s_\nu| \leqq k\varepsilon^\nu \qquad (1 \leqq \nu \leqq 2k-2)$$

and hence for $\nu = 1, 2, \ldots, 2k-2$ we have

$$|s_\nu|^2 \leqq 1 - \varepsilon^\nu + \left(k^2 + \frac{1}{4}\right)\varepsilon^{2\nu} < 1,$$

provided that

$$\left(k^2 + \frac{1}{4}\right)\varepsilon^\nu < 1.$$