ON ABELIAN GROUPS IN WHICH EVERY HOMOMORPHIC IMAGE CAN BE IMBEDDED

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§ 1. Introduction

In a previous paper [5][1] we have determined the structure of all abelian groups every subgroup of which is an endomorphic image of the group. Now we are concerned with the dual of this problem, namely, we describe all abelian groups G with

Property Q. *Every homomorphic image of G can be isomorphically imbedded in G.*

Otherwise expressed, G has property Q if to every factor group G/H of G there exists a subgroup F of G such that $F \simeq G/H$.

The problem of determining all groups with property Q has been stated in our cited paper [5]. Let us mention that the problem arising by the conjunction of these problems has been discussed in details in [2] and [3].

For the terminology and notations we refer to our paper [5].

The following lemmas will be made use of.

LEMMA 1. Any p-group G has a decomposition $G = G_1 + G_2$ where G_1 is a bounded group and rank $G_2 = \min_n$ rank $(p^n G) =$ final rank of G.

LEMMA 2. Each mixed group G has a decomposition $G = G_1 + G_2$ where G_1 is a torsion group whose p-components are bounded and G_2 is a mixed group with the following property: for each prime p_i, the p_i-component of the torsion subgroup of G_2 has a rank not exceeding any prescribed cardinal number $\mathfrak{m}_i$ with $\mathfrak{m}_i \geqq \max(\mathfrak{p}_i, \mathfrak{r}, \aleph_0)$. Here $\mathfrak{p}_i$ is the final rank of the p_i-component of the torsion subgroup of G and $\mathfrak{r}$ is the torsion free rank of G.

LEMMA 3. Each p-group G contains a basic subgroup B such that rank $(G/B) = \min_n$ rank $(p^n G)$.

[1] The numbers in brackets refer to the Bibliography given at the end of this paper.

LEMMA 4.[2] If G is a group with infinite torsion free rank $\mathfrak{r}$ and $\mathfrak{r} \leqq \mathfrak{p}_i$ holds for the rank $\mathfrak{p}_i$ of every p_i-component of the torsion subgroup of G, then G is isomorphic to some subgroup of the complete group[3]

$$\sum_{\mathfrak{r}} \mathfrak{R} + \sum_i \sum_{\mathfrak{p}_i} \mathfrak{C}(p_i^\infty).$$

LEMMA 5. Let J be a torsion free group of finite rank, S a serving subgroup of J and assume that all the elements of J, not contained in S, are of the same type τ. Then J is the direct sum of S and rational groups (of type τ) if (and only if) the elements $\neq 0$ of J/T have the same type τ for every serving subgroup T of J with $T \supseteq S$.

For the proofs of Lemmas 1, 2 and 4 see [3], for Lemma 3 we refer to [4] or [7], while Lemma 5 is a particular case of Theorem 8.4 in [1].

§ 2. Torsion groups with property Q

We begin with the description of the torsion groups with property Q. Since it is evident that a torsion group G has property Q if and only if every primary component of it has the same property, it follows that it suffices to consider only p-groups with property Q.

Our main result on p-groups is the following theorem.

THEOREM 1. *An abelian p-group G has property Q if and only if it contains a direct summand of the form*

$$\sum_{\mathfrak{m}} \mathfrak{C}(p^\infty) \tag{1}$$

where $\mathfrak{m}$ is the final rank of G, i. e.

$$\mathfrak{m} = \min_n \text{ rank } (p^n G).$$

Assume G is a p-group of property Q and $\mathfrak{m}$ is the final rank of G. Choose a basic subgroup B of G such that rank $(G/B) = \mathfrak{m}$ (cf. Lemma 3). Then G/B is isomorphic to (1) and therefore G contains a subgroup, and hence a direct summand isomorphic to (1).

Conversely, let G have a direct summand G_1 of the form (1). By Lemma 1 we decompose G in the form $G = G_2 + G_2^*$ where G_2 is a bounded group and rank $(G_2^*) = \mathfrak{m}$. Clearly, $G_1 \subseteq G_2^*$ and therefore we have $G_2^* = G_1 + G_3$.

[2] The statement of this Lemma holds true also when the inequality concerning the ranks is not assumed, but we do not need this strengthened statement.

[3] $\mathfrak{R}$ denotes the additive group of the rationals, $\mathfrak{C}(n)$ a cyclic group of order n where $1 \leqq n \leqq \infty$, while $\mathfrak{C}(p^\infty)$ is a quasicyclic group.