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AN ANALOGUE FOR RINGS OF A GROUP PROBLEM OF P. ERDŐS AND B. H. NEUMANN

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In response to a question posed by Erdős, B. H. Neumann [12] studied groups having no infinite subsets of pairwise non-commuting elements, which he characterized as those with center of finite index. Following Neumann's terminology, we define a ring R to be a Paul Erdős ring, or PE-ring, if every set of pairwise non-commuting elements is finite. We show that the center of every PE-ring is of finite index; and we establish several equivalent characterizations of PE rings, thereby obtaining an analogue of a recent theorem of Brodie and the third author [5, Theorem 2.1]. We proceed to discuss near-commutativity of PE-rings, in several different senses of near-commutativity; and in the final section of the paper, we discuss two special classes of PE-rings. In our study of near-commutativity, we encounter the problem of characterizing rings with finite commutator ideal, and we provide a solution.

Neumann's argument involves in a crucial way the class of FC-groups — that is, the class of groups in which each element has only finitely many conjugates, or equivalently the centralizer of each element has finite index. The latter notion is the one which makes sense for arbitrary rings, and we call a ring an FIC-ring if the centralizer of each element has finite index. Another notion which arises in the ring problem, but does not seem to have been extensively studied for groups, is that of coverings by finitely many centralizers of non-central elements; and we define a ring R to be an FCC-ring if R is either commutative or has such a covering.

1. Preliminaries

For elements $x, y \in R$, we denote by $[x,y]$ the commutator $xy - yx$; and we denote by $[R,R]$ the abelian group generated by all commutators. If X is an element or subset of R, we denote by $C(X)$, $A_r(X)$ and $A(X)$ respectively the centralizer of X, the right annihilator of X, and the two-sided annihilator of X. We denote the center of R by Z or $Z(R)$ and the commutator ideal

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by $C(R)$. For a subring $S \subsetneqq R$, we use the symbol $[R: S]$ for the index of S in R, by which we mean the index of $(S,+)$ in $(R,+)$.

We have given the definition of PE-ring in the introduction. For a positive integer n, we define an n-PE-ring to be a ring in which every set of pairwise non-commuting elements contains at most n elements. It is not immediately apparent whether every PE-ring is an n-PE-ring for some n; but that will turn out to be the case. We note that for groups, this was the original problem posed by Erdős; and Neumann obtained the solution as a consequence of his characterization of PE-groups.

Clearly, every non-commutative PE-ring possesses maximal non-empty finite subsets M of pairwise non-commuting elements such that every $y \in R \setminus M$ commutes with some $x \in M$. Hence, if $M = \{x_1, x_2, \ldots, x_r\}$ is such a set, then

$$R = C(x_1) \cup C(x_2) \cup \ldots \cup C(x_r). \tag{1}$$

In general, we say that R has a finite covering by centralizers — or that R is an FCC-ring — if R is either commutative or has a finite set $\{x_1, \ldots, x_r\}$ of non-central elements, not necessarily pairwise non-commuting, for which (1) holds. We have shown that the class of PE-rings is contained in the class of FCC-rings. In fact, the inclusion is proper, for if $R = R_1 \oplus R_2$, with R_1 finite and non-commutative and R_2 not an FCC-ring, then R is an FCC-ring covered by the centralizers of the elements $(y,0)$, where $y \in R_1 \setminus Z(R_1)$. Clearly, R is not a PE-ring, since R_2 is not. Incidentally, this example shows that the class of FCC-rings, unlike the class of PE-rings, fails to be closed under taking subrings and homomorphic images.

Obviously, all finite rings and all commutative rings are PE-rings; and so are rings of form $B \oplus C$, where B is a finite ring and C is a commutative ring. All R with $[R: Z]$ finite are PE-rings. The following example is a PE-ring which is indecomposable.

EXAMPLE 1.1. Let R be the algebra over $GF(p)$ with basis $\{x, y, z_1, z_2, z_3, \ldots\}$ and multiplication defined by $x^2 = x$, $xy = y$, $yx = x$, $y^2 = y$, $xz_i = z_i x = z_i$ and $yz_i = z_i y = z_i$ for $i = 1, 2, 3, \ldots$, and $z_i z_j = z_{i+j}$ for all $i, j = 1, 2, \ldots$. Then Z is the subalgebra generated by the z_i, which is in fact an ideal; and $[R: Z] = p^2$, so R is a PE-ring. It is not difficult to show that R is a $(p+1)$-PE-ring. We leave it to the reader to verify that R is indecomposable.

We conclude this section by stating some known results which will be crucial tools in our later work. Not surprisingly, the first is due to B. H. Neumann.

LEMMA 1.2 ([11], 4.4). *Let G be a group. If $G = H_1 \cup H_2 \cup \ldots \cup H_n$, where the H_i are subgroups, and we delete the H_i which are of infinite index in G, the union of the remaining subgroups is still equal to G.*