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UNIFORM BOUNDEDNESS BY AVERAGING

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Abstract. We consider a system of functional differential equations with infinite delay and derive conditions on Liapunov functionals to ensure that solutions are uniformly bounded and uniformly ultimately bounded. The analysis is based on the method of finding a bound on the average values of unknown solutions and Jensen's inequality. Comparisons between our theorems and those existing in the literature are also given.

1. Introduction

Consider a system of functional differential equations

$$x'(t) = F(t, x_t), \quad x(t) \in R^n \tag{1.1}$$

in which $F(t,\phi)$ is a functional defined for $t \geqq 0$ and $\phi \in C$, where C is the set of bounded continuous functions $\phi\colon (-\infty, 0] \to R^n$ with the supremum norm. For each $t \in R^+ = [0, +\infty)$, $C(t)$ denotes the set of continuous functions $\phi\colon [0,t] \to R^n$ with $\|\phi\| = \sup\left\{ \left|\phi(s)\right| : 0 \leqq s \leqq t\right\}$, where $|\cdot|$ is the Euclidean norm on R^n. We assume that for each $t_0 \geqq 0$ and each $\phi \in C(t_0)$ there is at least one solution $x(t, t_0, \phi)$ of (1.1) defined on an interval $[t_0, \alpha)$ with $x_{t_0} = \phi$. Here, $x_t(s) = x(t+s)$ for $s \leqq 0$. Moreover, if the solution remains bounded, then $\alpha = \infty$.

We are interested in conditions on Liapunov functionals which will ensure that solutions are uniformly bounded (UB) and uniformly ultimately bounded (UUB). Much discussion of these concepts and of the above mentioned existence properties may be found in Burton [3] or Yoshizawa [15], for example.

Definition 1.1. Solutions of (1.1) are UB if for each $B_1 > 0$ there exists $B_2 > 0$ such that $\left[t_0 \geqq 0,\ \phi \in C,\ \|\phi\| < B_1,\ t \geqq t_0\right]$ imply that $\left|x(t, t_0, \phi)\right| \leqq B_2$.

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DEFINITION 1.2. Solutions of (1.1) are UUB if there is a $B > 0$ and for each $B_3 > 0$ there is a $T > 0$ such that $\left[t_0 \geqq 0,\, \phi \in C,\, \|\phi\| < B_3,\, t \geqq t_0 + T\right]$ imply that $\left|x(t, t_0, \phi)\right| < B$.

These are generalizations of uniform stability and uniform asymptotic stability. An example will bring our work into focus.

EXAMPLE 1.1. Let $D : [0, \infty) \to [0, 1)$ be continuous, $L^1[0, \infty)$, and let $\int_t^\infty D(u)\,du \in L^1[0, \infty)$. Suppose that $a, b : [0, \infty) \to [0, \infty)$ are continuous, that n is the quotient of odd positive integers, and that $p : [0, \infty) \to R$ is bounded and continuous. Finally, suppose that there is an $L > 0$ such that

$$-a(t) + \int_0^\infty D(s)\,ds + \left[b^2(t)/(2L)\right] \leqq 0. \tag{1.2}$$

Note that $p(t)$ is bounded, but we will be most interested in the case in which $b(t)$ ranges from zero to infinity.

Consider the scalar equation

$$x' = -a(t)x^3 - x^n + \int_0^t D(t-s)x^3(s)\,ds + b(t)p(t) \tag{1.3}$$

and define a Liapunov functional V by

$$V(t, x_t) = (1/4)x^4 + (1/2)\int_0^t \int_{t-s}^\infty D(u)\,du\,x^6(s)\,ds \tag{1.4}$$

so that along a solution of (1.3) we obtain

$$\begin{aligned} V'(t, x_t) &= -a(t)x^6 - x^{n+3} + x^3 \int_0^t D(t-s)x^3(s)\,ds + x^3 b(t)p(t) \\ &\quad + (1/2)\int_0^\infty D(u)\,du\,x^6 - (1/2)\int_0^t D(t-s)x^6(s)\,ds \\ &\leqq -a(t)x^6 - x^{n+3} + (1/2)\int_0^t D(t-s)\big(x^6(t) + x^6(s)\big)\,ds + (1/2L)x^6 b^2(t) \\ &\quad + (L/2)p^2(t) + (1/2)\int_0^\infty D(u)\,du\,x^6 - (1/2)\int_0^t D(t-s)x^6(s)\,ds \\ &= \left[-a(t) + (1/2)\int_0^t D(s)\,ds + (1/2)\int_0^\infty D(u)\,du + (1/2L)b^2(t)\right]x^6 \\ &\quad - x^{n+3} + (L/2)p^2(t) \end{aligned}$$