Acta Math. Hungar.
95 (1–2) (2002), 1–20.

DISTRIBUTION MODULO 1 OF SOME OSCILLATING SEQUENCES. III

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Abstract. For some oscillating functions, such as $h(x) = x^\pi \log^3 x \cos x$, we consider the distribution properties modulo 1 (density, uniform distribution) of the sequence $h(n)$, $n \geqq 1$. We obtain positive and negative results covering the case when the factor $x^\pi \log^3 x$ is replaced by an arbitrary function f of at most polynomial growth belonging to any Hardy field. (The latter condition may be viewed as a regularity growth condition on f.) Similar results are obtained for the subsequence $h(p)$, taken over the primes $p = 2, 3, 5, \ldots$.

1. Introduction

In the theory of distribution modulo 1, the basic problem is, given a sequence of real numbers, to determine its behaviour modulo 1. In many cases, the sequence is defined in a natural way, namely as the sequence $\big(h(n)\big)_{n=1}^{\infty}$ of values taken by a real-valued function $h(x)$ at the positive integer points. The celebrated equidistribution theorem of Weyl asserts that $\big(h(n)\big)$ is uniformly distributed modulo 1 (henceforward — u.d.) if the function $h(x)$ is a polynomial with at least one irrational coefficient (besides the free term).

Some results along these lines were obtained also for more general functions; see, for example, [12]. In [4], necessary and sufficient conditions for uniform distribution and for density modulo 1 of the sequences $\big(h(n)\big)$ were provided for functions h growing at most polynomially and belonging to a certain large class U (the union of all Hardy fields; see Section 2 *infra*). The condition $h \in U$ may be viewed as a monotonicity (or regular growth) condition on h. The class U contains functions defined by "natural" formulas, say $\int \frac{x^a}{\log^b x}\,dx$.

The case when $h(x)$ is oscillating (no monotonicity conditions) is more difficult to deal with. A metrical result, due to LeVeque [13], asserts that

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[1] Research supported in part by the Israel Science Ministry, the Israeli Research Foundation (Grant #216/98) and the Bi-National Israel–USA Science Foundation.

[2] Research supported in part by NSF grant # DMS-9971120 and the Bi-National Israel–USA Science Foundation.

Key words and phrases: Density modulo 1, distribution modulo 1, uniform distribution, exponential sums, Hardy field.

2000 Mathematics Subject Classification: primary 11K06, 11K38, 11J71; secondary 11L07, 11L20.

for any increasing sequence a_n of integers the sequence $a_n \cos a_n\alpha$ is u.d. for almost every α (in the sense of Lebesgue measure). Of course, this is not the case in general (i.e., for an arbitrary sequence a_n) for every α. One may inquire whether for some special sequences a_n the metrical result can be replaced by a global result. Furstenberg and Weiss [9] proved that for almost every α the set of solutions n of the inequality $\|n \cos n\alpha\| < \varepsilon$ is not of bounded gaps for every $\varepsilon < \frac{1}{2}$ (unlike the case of polynomials, where the set of solutions n of the inequality $\big\| P(n) - P(0) \big\| < \varepsilon$ is of bounded gaps for every $\varepsilon > 0$). They raised the question whether this inequality has a solution for every α and $\varepsilon > 0$.

This question was settled in [1], with further refinements obtained in [5] and [6]. It turns out that, not only does the inequality necessarily have solutions, but actually the sequence $(n \cos n\alpha)$ is necessarily u.d. unless α is a rational multiple of π. Moreover, the uniform distribution results of these papers apply to sequences of the form $P(n)g(n\alpha)$, where P is an arbitrary polynomial and g a periodic "highly differentiable" function. Weaker conclusions, namely density modulo 1 or small value results, were obtained for more general sequences.

In this paper we continue this line of study. First, we consider the sequence $f(n)g(n\alpha)$ with a "nice" function f which is not necessarily a polynomial. We are able to obtain general density and uniform distribution results for such sequences. Next, we look at the behaviour of the subsequence of $P(n)g(n\alpha)$ obtained by restricting n to the primes. Again, we prove a uniform distribution result under some conditions. Note that already the uniform distribution of the sequence $p\alpha$, where p ranges over the primes, is a non-trivial result due to Vinogradov [16, pp. 177–180] (see also [8, p. 47] and [15]).

Our results assume a particularly neat formulation under the additional assumption that $f(x)$ belongs to some Hardy field and its growth is at most polynomial. (This assumption is similar to the one imposed in [4]; see the introduction to that paper.) Under this assumption, complicated conditions imposed on f and its derivatives may be replaced by simple conditions on the growth of f itself, and the results take an especially short and transparent form. (Compare, for example, Theorem 2.2 with its technical counterpart, Theorem 3.2.) Nevertheless, the results remain quite general and interesting due to the fact that the class U, defined as the union of all Hardy fields, is rich enough. (In particular, it contains solutions of numerous differential and difference equations, as well as functions defined by a large class of "elementary" formulas.) Note that the assumption for functions to belong to some Hardy field is a natural setting in some other problems (see [7], [4]).

Some background information on Hardy fields is presented at the beginning of Section 2. To demonstrate the broad range of sequences our results apply to, we proceed with listing our qualitative results under the Hardy field assumption, $f \in U$, accompanied by the appropriate examples. In Sec-