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ALGEBRAS GENERATED BY SEMICENTRAL IDEMPOTENTS

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Abstract. Let R be a unital K-algebra, where K is a commutative ring with unity. An idempotent $e \in R$ is *left semicentral* if $Re = eRe$, and R is *SCI-generated* if it is generated as a K-module by left semicentral idempotents. This paper develops the basic properties of SCI-generated algebras and characterizes those that are also prime, semiprime, primitive, or subdirectly irreducible. Minimal ideals and the socle of SCI-generated algebras are investigated. Conditions are found to describe a large class of SCI-generated algebras via generalized triangular matrix representations. SCI-generated piecewise domains are characterized. Examples are given that illustrate the breadth and diversity of the class of SCI-generated algebras.

0. Introduction

Throughout this paper K will denote a commutative ring with unity, R a K-algebra with unity, and W a K-algebra not necessarily having unity. All K-modules are unitary. An idempotent $e \in W$ is *left* (*right*) *semicentral* in W if $We = eWe$ (respectively, $eW = eWe$), [4, p. 569]. Observe that the right (left) semicentral idempotents are exactly the elements that give rise to left (right) multiplication mappings which are idempotent K-algebra endomorphisms. We use $\mathcal{S}_\ell(W)$ and $\mathcal{S}_r(W)$ for the sets of all left and right semicentral idempotents, respectively. Each of these sets is multiplicatively closed and $\mathcal{S}_\ell(W) \cap \mathcal{S}_r(W) = \mathbf{B}(W)$, the set of all central idempotents in W. We say R is *SCI-generated* if $R = \mathrm{Mod}_K\big(\mathcal{S}_\ell(R)\big)$, where $\mathrm{Mod}_K(X)$ denotes the K-module generated by X. We will show that the SCI-generated condition is left-right symmetric, i.e., R is SCI-generated if and only if $R = \mathrm{Mod}_K\big(\mathcal{S}_r(R)\big)$. Observe that every ring T with unity contains the SCI-generated algebra $\mathrm{Mod}_C\big(\mathcal{S}_\ell(T)\big)$, where C is the center of T.

The class of SCI-generated algebras is broad and diverse, as we will illustrate by examples. It of course contains all algebras generated by central idempotents. Such algebras have received considerable attention, e.g., see [2], [16], and [18]. Also SCI-generated algebras form an important subclass

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of the class of algebras which are both LSD-generated and RSD-generated, a point that will be elaborated on in the next section.

This paper develops structure theory for SCI-generated algebras, especially unital ones. Prime, semiprime, primitive, and subdirectly irreducible SCI-generated unital algebras are classified, as are the ideals associated with them. Minimal ideals and the socle are investigated. Special results accrue under mild chain conditions. Applications are made to algebras with a complete set of triangulating idempotents and to quasi-Baer rings. This leads to generalized triangular matrix representations. Moreover, connections are made to piecewise domains.

Throughout this paper $\mathbf{I}(W)$ denotes the set of idempotent elements of W. Terminology such as homomorphism, ideal, prime, and simple, etc. will refer to those concepts in the category of K-algebras. The term "ideal" means a two-sided ideal.

1. Preliminaries and examples

We begin with a useful, albeit technical lemma, the proof of which can be found in [10, Proposition 1.1].

LEMMA 1.1. *Let $e \in \mathbf{I}(R)$. Then the following conditions are equivalent:*
(i) $e \in \mathcal{S}_\ell(R)$;
(ii) $1 - e \in \mathcal{S}_r(R)$;
(iii) $xe = exe$, *for each* $x \in R$;
(iv) $(1-e)Re = 0$;
(v) $(1-e)x = (1-e)x(1-e)$, *for each* $x \in R$;
(vi) eR *is an ideal of* R;
(vii) $R(1-e)$ *is an ideal of* R;
(viii) $eR(1-e)$ *is an ideal of* R *and* $eR = eR(1-e) \oplus Re$, *as a direct sum of left ideals;*
(ix) *the function defined by* $\phi(x) = \begin{pmatrix} exe & ex(1-e) \\ 0 & (1-e)x(1-e) \end{pmatrix}$ *is a K-algebra isomorphism from R onto* $\begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}$;
(x) $\{a \in R \mid Rea \subseteqq Re\} = \{c \in R \mid ce = ec\}$;
(xi) $\ell(Re) = \ell(e)$, *where* $\ell(-)$ *is the left annihilator.*

Observe from Lemma 1.1 that if $e \in \mathbf{I}(R)$, then e is a left (right) semicentral idempotent in the idealizer of $eR\,(Re)$; and if R is semiprime, then $\mathcal{S}_\ell(R) = \mathbf{B}(R) = \mathcal{S}_r(R)$. Further properties can be found in [10]. Moreover, from the equivalence of (i) and (ii) of Lemma 1.1, it follows that the SCI-generated condition is left-right symmetric (i.e., $R = \mathrm{Mod}_K\left(\mathcal{S}_\ell(R)\right)$ if and only if $R = \mathrm{Mod}_K\left(\mathcal{S}_r(R)\right)$).