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ON INVERSION OF BESSEL POTENTIALS ASSOCIATED WITH THE LAPLACE–BESSEL DIFFERENTIAL OPERATOR

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Abstract. Explicit inversion formulas of Balakrishnan–Rubin type and a characterization of Bessel potentials associated with the Laplace–Bessel differential operator $\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial^2}{\partial x_n^2} \right)$ $(\nu > 0)$ are obtained. As an auxiliary tool the B-metaharmonic semigroup is introduced and some of its properties are investigated.

Introduction

The classical Bessel potentials, an important technical tool in harmonic analysis, theory of functions and partial differential equations, are defined in terms of Fourier transforms by

$$(J^\alpha \varphi)^\wedge(x) = \left(1 + |x|^2\right)^{-\alpha/2} \varphi^\wedge(x) \quad (\alpha > 0, x \in R^n).$$

These potentials are interpreted as the negative fractional powers of the differential operator $(I - \Delta)$ (I, identity operator and Δ, Laplacian). The Bessel potentials and Liouville spaces generated by them, have been investigated in the fundamental papers by N. Aronszajn and K. Smith [5] and A. Calderon [7] in 1961. Further generalizations and applications were developed by E. Stein, F. Mulla, P. Szeptycki, R. Adams, P. Lizorkin, T. Flett and other mathematicians. (For details see the monographs E. Stein [25] and S. Samko, A. Kilbas, A. Marichev [23].)

An important trend in potential theory is to elaborate some constructions to determine the inverse of potential operators. Hypersingular integral theory has appeared as a result of the investigations of E. Stein [27], P. Lizorkin [14], R. Wheeden [30], S. Samko [24], B. Rubin [21], V. Nogin [19] and others in this area (for further information see [22] and [23] and references therein).

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The singular integrals, potentials and related topics associated with the Laplace–Bessel differential operator

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left(\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n} \right) \quad (\nu > 0),$$

which is known as an important differential operator in analysis and its applications, have been the research areas for many mathematicians such as B. Muckenhoupt and E. Stein [17], I. Kipriyanov and M. Klyuchantsev [10–12], K. Trimeche [29], L. Lyakhov [16], K. Stempak [28], A. D. Gadjiev and I. A. Aliev [1–3], [8], V. Guliev [9] and others.

In this paper an inversion formula for Bessel potentials associated with the Laplace–Bessel differential operator Δ_B is obtained. Moreover, to express a function as a Bessel potential some sufficient conditions are given. To do this, as an auxiliary technical tool the B-metaharmonic semigroup is defined and some of its properties are investigated.

We recall that the useful idea to invert the classical Bessel, Riesz and parabolic potentials by means of various semigroups is due to B. S. Rubin [21–22].

The structure of the paper is as follows. In Section 1 we present some definitions and auxiliary results. In Section 2 we introduce the B-metaharmonic semigroup and give its properties we need. The connection between this semigroup and the Bessel potentials generated by generalized translation operators is established in Section 3. The main results of the paper (Theorems 1 and 2) are formulated and proved in Section 4.

1. Definitions, notation and preliminaries

Let $R_n^+ = \{ x \in R_n,\, x = (x_1, \ldots, x_n),\, x_n > 0 \}$ and $S^+ = S^+(R_n^+)$ be the Schwartz space of infinitely differentiable and rapidly decreasing functions, even in x_n. The dual space of S^+ is denoted by $(S^+)'$. C^0 will denote the space of all continuous functions, vanishing at infinity.

For a fixed parameter $\nu > 0$, let $L_{p,\nu} = L_{p,\nu}(R_n^+)$ be the space of measurable functions with a finite norm

$$\|f\|_{p,\nu} = \left(\int\limits_{R_n^+} |f(x)|^p x_n^{2\nu}\, dx \right)^{1/p}, \quad 1 \leqq p < \infty.$$