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PRODUCTS OF COMPACTIFICATIONS

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Abstract. A product operation of compactifications is defined and its different properties are studied. Some applications are considered.

Let X be a completely regular (Hausdorff) topological space and let b_1X and b_2X be two compactifications of X, i.e. two compact spaces with some inclusions $i_1 : X \subset b_1X$, $i_2 : X \subset b_1X$ with the images $i_1(X)$, $i_2(X)$ being dense in b_1X and b_2X, respectively.

The inclusions i_1, i_2 define a new inclusion $i : X \subset b_1X \times b_2X$ of X in the direct product of b_1X and b_2X by the relation $i(x) = \big(i_1(x), i_2(x)\big) \in b_1X \times b_2X$ for all $x \in X$. The closure of $i(X) \subset b_1X \times b_2X$ is some new compactification bX of X. Throughout this article we denote this compactification by $(b_1 \times b_2)X$ or simply $b = b_1 \times b_2$.

Definition. The compactification bX will be called the product of the compactifications b_1 and b_2 of the space $X : bX = (b_1 \times b_2)X$ or $b = b_1 \times b_2$.

In this article different properties of this product operation as well as of the supplements $(b_1 \times b_2)X \backslash X$ in compactification products are studied. Infinite products and some applications are also considered.

A compactification aX of X is said [4] to follow another bX if there exists some continuous mapping $\pi_b^a : aX \to bX$ such that $\pi_b^a(x) = x$ for any point $x \in X = i_a(X) = i_b(X)$. Here i_a,i_b are the inclusions of X in aX and bX, respectively. Throughout this article the symbol $bX \leqq aX$ means that the compactification aX follows bX. Since $\pi_b^a(aX)$ is a compact subspace of bX containing X and X is dense in bX then $\pi_b^a(aX) = bX$, i.e. π_b^a is always a surjection.

The inclusion $i : X \subset b_1X \times b_2X$ is the composition $i = (i_1 \times i_2)\Delta$ where Δ is the diagonal inclusion of X in the Décartes square $X \times X$, $\Delta(x) = (x, x) \in X \times X$, and $i_1 \times i_2$ is the natural inclusion $X \times X \subset b_1X \times b_2X$. Hence the compositions of i with the natural projections $\pi_j : b_1X \times b_2X \to b_jX$, $j = 1, 2$, are homeomorphisms of X. Consequently the restrictions of the projections π_j to $(b_1 \times b_2)X \subset b_1X \times b_2X$ define the mappings of compact-

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ifications $(b_1 \times b_2)X \to b_jX$, $j = 1, 2$. Thus the compactification $(b_1 \times b_2)X$ always follows b_1X and b_2X.

PROPOSITION 1. *The compactification $b_1 \times b_2$ is minimal among all compactifications aX of X following both compactifications b_1X and b_2X.*

PROOF. Let a compactification aX follow b_1X and b_2X. Since $(b_1 \times b_2)X$ follows b_1X and b_2X too, to prove the proposition it is sufficient to show that the compactification aX follows $(b_1 \times b_2)X$.

The mappings $\pi^a_{b_j} : aX \to b_jX$, $j = 1, 2$, define a mapping $\alpha : aX \to b_1X \times b_2X$ by $\alpha(\xi) = \left(\pi^a_{b_1}(\xi), \pi^a_{b_2}(\xi)\right)$ for all points $\xi \in aX$. The restriction α to $X \subset aX$ coincides with $(i_1 \times i_2)\Delta = i$, hence $iX \subset \alpha(aX)$. Since $\alpha(aX)$ is compact and $iX = X$ is dense in $(b_1 \times b_2)X \subset b_1X \times b_2X$ then $\alpha(aX) = (b_1 \times b_2)X$. So α maps the compactification aX on $(b_1 \times b_2)X$ and aX follows $(b_1 \times b_2)X$. The proposition is proved.

So the product operation is commutative: $(b_1 \times b_2)X = (b_2 \times b_1)X$. Furthermore, $(b_1 \times b_2)X = b_2X$ if and only if $b_1X \leqq b_2X$. In particular, the product operation is idempotent, i.e., $(b \times b)X = bX$ (or simply $b \times b = b$) for any compactification bX of a space X.

PROPOSITION 2 (the associativity property). *For any three compactifications b_1X, b_2X and b_3X of X the following relation is true:*

$$\left((b_1 \times b_2) \times b_3\right) X = \left(b_1 \times (b_2 \times b_3)\right) X$$

$$\left(\textit{or simply} \quad (b_1 \times b_2) \times b_3 = b_1 \times (b_2 \times b_3)\right).$$

PROOF. The imbeddings $i_j : X \subset b_jX$ define an imbedding $i : X \subset b_1X \times b_2X \times b_3X$ of X in the direct product of these compactifications by the relation $i(x) = \left(i_1(x), i_2(x), i_3(x)\right)$ for any point $x \in X$. Let $(b_1 \times b_2 \times b_3)X$ be the closure of $X = i(X)$ in $b_1X \times b_2X \times b_3X$. This is some compactification of X. We will denote it also as $b_1 \times b_2 \times b_3$. As in Proposition 1, this compactification is minimal among all compactifications succeeding b_1X, b_2X and b_3X.

To prove this proposition it is sufficient to show that each compactification in it is equal to $b_1 \times b_2 \times b_3$. We do this for the compactification $(b_1 \times b_2) \times b_3$.

The compactification $b_1 \times b_2 \times b_3$ of X succeeds b_1X and b_2X and in accordance with Proposition 1 it succeeds $(b_1 \times b_2)X$. Since $b_1 \times b_2 \times b_3$ succeeds also b_3X according to Proposition 1 we have also $b_1 \times b_2 \times b_3 \geqq (b_1 \times b_2) \times b_3$.

On the other hand the compactification $(b_1 \times b_2) \times b_3$ succeeds $(b_1 \times b_2)X$ and b_3X (Proposition 1). Since $b_1 \times b_2$ succeeds b_1X and b_2X (Proposition 1) $(b_1 \times b_2) \times b_3$ also succeeds b_1X and b_2X. Since $b_1 \times b_2 \times b_3$ is minimal among all such compactifications then $b_1 \times b_2 \times b_3 \leqq (b_1 \times b_2) \times b_3$. Thus $(b_1 \times b_2) \times b_3 = b_1 \times b_2 \times b_3$. Similar arguments also provide the equation $b_1 \times (b_2 \times b_3) = b_1 \times b_2 \times b_3$. The proposition is proved.