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MODULES WHICH SATISFY THE RADICAL FORMULA

D. PUSAT-YILMAZ (Bolu) and P. F. SMITH (Glasgow)

Abstract. Let R be a commutative ring. Then an R-module M satisfies the radical formula when $M = M_1 \oplus M_2$ is a direct sum of a submodule M_1 which satisfies the radical formula and a semi-artinian submodule M_2.

1. Introduction

Throughout this note all rings are commutative with identity and all modules are unital modules. Let R be a ring and let M be an R-module. A submodule K of M is called *prime* if $K \neq M$ and whenever $r \in R$, $m \in M$ with $rm \in K$ then $m \in K$ or $rM \subseteqq K$. Given a proper submodule N of M, the (*prime*) *radical* $\operatorname{Rad}_M(N)$ of N (*in* M) is the intersection of all prime submodules of M containing N, or $\operatorname{Rad}_M(N) = M$ in case no prime submodule of M contains N. The radical of submodules has been extensively studied in recent years (see, for example, [3]–[6], [8], [9]).

Let N be a submodule of a module M. Then the *envelope* $E_M(N)$ *of* N (in M) is the set of elements rm of M such that $r \in R$, $m \in M$ and $r^n m \in N$ for some positive integer n. In general, $0 \in E_M(N)$ but $E_M(N)$ is not a submodule of M, for a given submodule N. We denote by $RE_M(N)$ the submodule of M generated by the set $E_M(N)$. It is easy to check that $RE_M(N)$ consists of all finite sums of elements in $E_M(N)$ and that

$$N \subseteqq RE_M(N) \subseteqq \operatorname{Rad}_M(N)$$

for any submodule N of M. Following [5] we say that the module M *satisfies the radical formula* in case $\operatorname{Rad}_M(N) = RE_M(N)$ for every submodule N.

Lemma 1.1 ([5, Theorem 1.5]). *If a module M satisfies the radical formula then so too does any homomorphic image of M.*

Corollary 1.2. *Any cyclic module satisfies the radical formula.*

Proof. By Lemma 1.1.

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In contrast to Corollary 1.2, Leung and Man [4, Corollary 2.3] prove that if R is a Noetherian domain then the R-module $R \oplus R$ satisfies the radical formula only if R is a Dedekind domain. Moreover, for any Noetherian ring R, every R-module satisfies the radical formula if and only if the R-module $R \oplus R$ satisfies the radical formula (see [4, Corollary 5.2]). In addition, in [4, Theorem 1.1] necessary and sufficient conditions are given for a Noetherian ring R to have the property that every R-module satisfies the radical formula. This follows earlier work of Jenkins and Smith [3].

It is proved in [4, Theorem 3.5] (see also [9, Corollary 2.9]) that if R is an Artinian ring then every R-module satisfies the radical formula. More generally, Sharif, Sharifi and Namazi [9, Theorem 2.8] prove that if R is a ring such that every prime ideal is maximal then every R-module satisfies the radical formula. The purpose of this note is to generalise this result of Sharif, Sharifi and Namazi.

Let R be any ring. Given a maximal ideal $\mathbf{m}$ of R, an R-module M will be called $\mathbf{m}$-*special* if for each $a \in \mathbf{m}$, $m \in M$, there exist a positive integer n and an element c in $R \backslash \mathbf{m}$ such that $ca^n m = 0$. Moreover, the R-module M is called *special* if M is $\mathbf{m}$-special for every maximal ideal $\mathbf{m}$ of R. The ring R has the property that every prime ideal is maximal if and only if every R-module is special (see Theorem 3.5).

Our generalization of [9, Theorem 2.8] is as follows: given any ring R, any R-module M' which satisfies the radical formula and any special R-module M'', then the R-module $M' \oplus M''$ also satisfies the radical formula (Theorem 4.8). We also prove that if R is a domain, M' an R-module which satisfies the radical formula and M'' a divisible R-module, then the R-module $M' \oplus M''$ satisfies the radical formula (Theorem 2.2).

Note that if R is any ring and an R-module $M = M' \oplus M''$ is a direct sum of submodules M', M'', then M satisfies the radical formula only if M' and M'' both satisfy the radical formula by Lemma 1.1.

We denote the set of positive integers by $\mathbf{N}$. Any unexplained terminology can be found in [1].

2. Divisible modules

We begin this section with the following simple lemma which is included for completeness.

LEMMA 2.1. *Let $\phi : M \to M'$ be an epimorphism, for given modules M, M', and let N be a submodule of M. Then*

$$\phi\big(\operatorname{Rad}_M(N)\big) \subseteqq \operatorname{Rad}_{M'}\big(\phi(N)\big).$$

PROOF. Let K denote the kernel of ϕ. Then

$$\phi\big(\operatorname{Rad}_M(N)\big) \subseteqq \phi\big(\operatorname{Rad}_M(N+K)\big) = \operatorname{Rad}_{M'}\big(\phi(N)\big)$$