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THE ATOMIC DECOMPOSITION OF MOLECULE ON DISCRETE HARDY SPACES

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Abstract. We show the atomic decomposition of molecule on discrete Hardy spaces without using the theory of continuous Hardy spaces.

1. Introduction

Boza and Carro [1] obtained the atomic decomposition characterization of the discrete Hardy spaces $H^p(\mathbf{Z})$. Kanjin and Satake [2] defined discrete molecules and they proved that molecules belong to $H^p(\mathbf{Z})$ by using the theory of continuous Hardy spaces $H^p(\mathbf{R})$.

In this paper we show that every molecule can be decomposed into atoms without using the theory of continuous Hardy spaces $H^p(\mathbf{R})$.

2. Definitions

Let $l^p(\mathbf{Z})$, $0 < p \leqq \infty$ be the space of sequences $c = \{c(n)\}_{n \in \mathbf{Z}}$ such that

$$\|c\|_p = \left(\sum_{n \in \mathbf{Z}} |c(n)|^p \right)^{1/p} < \infty \ (0 < p < \infty), \quad \|c\|_\infty = \sup_{n \in \mathbf{Z}} |c(n)| < \infty.$$

The discrete Hardy spaces $H^p(\mathbf{Z})$, $0 < p < \infty$ are the spaces of sequences $c = \{c(n)\}_{n \in \mathbf{Z}}$ such that

$$\|c\|_{H^p} = \|c\|_p + \|H^d c\|_p < \infty,$$

where $H^d c$ is the discrete Hilbert transform of c given by

$$H^d c(n) = \sum_{l \neq n} \frac{c(l)}{n - l}, \quad n \in \mathbf{Z}.$$

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We define atoms on $H^p(\mathbf{Z})$ (see [1]).

DEFINITION 1. Let $0 < p \leqq 1 \leqq q \leqq \infty$, $p < q$ and k be a non negative integer. We say that a sequence $a = \{a(n)\}_{n\in\mathbf{Z}}$ is a (p,q,k)-atom if there exist integers $n_0 < m_0$ which satisfy the conditions:

$$\text{(i)} \qquad \operatorname{supp} a \subset \{n_0, n_0+1, \dots, m_0-1, m_0\},$$

$$\text{(ii)} \qquad \|a\|_q \leqq (m_0 - n_0 + 1)^{1/q-1/p},$$

$$\text{(iii)} \qquad \sum_{n=-\infty}^{\infty} n^j a(n) = 0 \quad \text{for} \quad j = 0, 1, \dots, k.$$

The atomic decomposition characterization of discrete Hardy spaces is as follows.

THEOREM 1 [1]. *Let $0 < p \leqq 1 \leqq q \leqq \infty$, $p < q$ and $L = [1/p - 1]$. Let k be an integer such that $k \geqq L$.*

(1) *There exists a constant K such that if a is a (p,q,k)-atom then $\|a\|_{H^p} \leqq K$.*

(2) *There exists a constant K such that if $c \in H^p(\mathbf{Z})$ then there are a sequence $\{a_j\}$ of (p,∞,k)-atoms a_j and a sequence $\{\lambda_j\}$ of real numbers λ_j with $\sum |\lambda_j|^p \leqq K\|c\|_{H^p}$ satisfying $c = \sum \lambda_j a_j$.*

3. Molecules

We define molecules on $H^p(\mathbf{Z})$ (see [2]).

DEFINITION 2. Let $0 < p \leqq 1 \leqq q < \infty$, $p < q$, $\alpha > q/p - 1$. We say that a sequence $M = \{M(n)\}_{n\in\mathbf{Z}}$ is a (p,q,α)-molecule (centered at n_0) if there exist positive integer r and $n_0 \in \mathbf{Z}$ which satisfy

$$\sum_{|n-n_0|\leqq 2r} |M(n)|^q \leqq r^{1-q/p}, \tag{3}$$

$$\sum_{|n-n_0|\geqq 2r} |M(n)|^q |n-n_0|^\alpha \leqq r^{\alpha+1-q/p}, \tag{4}$$

$$\sum_{n=-\infty}^{\infty} n^j M(n) = 0, \; j = 0, 1, \dots, L = [1/p - 1]. \tag{5}$$

REMARK. Our definition of molecules is slightly different from the one defined in [2]. However, we can prove H^p boundedness of singular integrals by using our molecules (see §4).