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ALGEBRAS BETWEEN $C^*(X)$ AND $C(X)$ THAT ARE CLOSED UNDER COUNTABLE COMPOSITION

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Abstract. Let $C(X)$ be the algebra of all real-valued continuous functions on a completely regular space X, and $C^*(X)$ the subalgebra of bounded functions. We show that the space of real maximal ideals of an intermediate algebra between $C^*(X)$ and $C(X)$ is a C^*-embedded closed subspace of the product of βX with a product of copies of the real line. We make use of this embedding to provide a new characterization of the intermediate algebras that are closed under countable composition, and to exhibit an example of an intermediate algebra on $\mathbf{N}$ that is closed under countable composition but not isomorphic to any $C(T)$.

Introduction

Let $C(X)$ be the algebra of all real-valued continuous functions on a nonempty completely regular space X, and $C^*(X)$ the subalgebra of bounded functions. This paper deals with subalgebras of $C(X)$ containing $C^*(X)$. We shall refer to them as *intermediate algebras on* X. They are uniformly closed Φ-algebras in the sense of Henriksen–Johnson. We shall restrict our attention to those intermediate algebras A that are closed under countable composition (i.e., if (f_n) is a sequence from A, and $g \in C(\mathbf{R}^{\mathbf{N}})$, then the composition $g \circ (f_1, f_2, \ldots)$ is in A). It is known that, under this hypothesis, A is a homomorphic image of some $C(T)$.

It is not easy to exhibit examples of intermediate algebras that are closed under countable composition but not isomorphic to any $C(T)$. The existence of such an algebra is equivalent to the existence of a realcompact space with a closed subspace that is C^*-embedded but not C-embedded. Apart from the example in [5, 3.4] we know of no other in the literature. Let βX be the Stone–Čech compactification of X. We shall show that the space of all real maximal ideals of an intermediate algebra on X is a C^*-embedded closed subspace of the product of βX with a product of copies of the real line. We make use of this embedding to provide a new characterization of the intermediate algebras that are closed under countable composition, and to

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give a new (and perhaps more natural) example of an intermediate algebra that is closed under countable composition but not isomorphic to any $C(T)$.

1. Preliminaries

We shall basically adhere to the notation and terminology in [4]. Throughout the paper X will be a nonempty completely regular Hausdorff space, and βX will denote the Stone–Čech compactification of X.

For any intermediate space Y between X and βX, the restriction morphism from $C(Y)$ to $C(X)$, which sends $g \in C(Y)$ to $g|_X$, is clearly injective. We shall always consider $C(Y)$ as an intermediate algebra on X.

As usual, $Z(f)$ will denote the *zero-set* of $f \in C(X)$, i.e., $Z(f) = \big\{x \in X : f(x) = 0\big\}$. For $f \in C^*(X)$, f^β will denote the continuous extension of f to βX.

If the function $f \in C(X)$ is regarded as a continuous mapping of X into the one-point compactification $\mathbf{R}^* = \mathbf{R} \cup \{\infty\}$ of $\mathbf{R}$, it has an extension $f^* : \beta X \to \mathbf{R}^*$. The set of points in βX where f^* takes real values is denoted by $\upsilon_f X$, i.e.,

$$\upsilon_f X = \big\{p \in \beta X : f^*(p) \neq \infty\big\}.$$

The space $\upsilon_f X$ is locally compact and σ-compact, hence realcompact, and it is the largest subspace of βX to which f can be continuously extended.

For $F \subseteqq C(X)$, we shall write

$$\upsilon_F X = \bigcap \{\upsilon_f X : f \in F\}.$$

The space $\upsilon_F X$ is realcompact, and certainly $F \subseteqq C(\upsilon_F X)$. With this notation, $\upsilon_C X = \upsilon X$ (the Hewitt realcompactification of X) and $\upsilon_{C^*} X = \beta X$.

A maximal ideal M of an intermediate algebra A is said to be real if the residue-class field A/M is isomorphic to $\mathbf{R}$. If we identify βX with the maximal ideal space of A, then $\upsilon_A X$ is the space of all real maximal ideals of A (see [2, 4.3]). The intermediate algebra A is isomorphic to $C(T)$, for some topological space T, if and only if $A = C(\upsilon_A X)$.

2. The embedding of $\upsilon_F X$ in $\beta X \times \mathbf{R}^F$

Let $F \subseteqq C(X)$. We shall write $e_F : X \to \mathbf{R}^F$ to denote the evaluation mapping induced by F, i.e., $\pi_f \circ e_F = f$, for any $f \in F$, where π_f is the f-th coordinate projection of $\mathbf{R}^F$ onto $\mathbf{R}$. We shall write $e_F^* : \upsilon_F X \to \mathbf{R}^F$ to denote the continuous extension of e_F to $\upsilon_F X$. Finally, we shall denote by υ_F the embedding $X \to \beta X \times \mathbf{R}^F$ defined by $\upsilon_F = (\beta, e_F)$, where $\beta : X \to \beta X$ is the inclusion mapping.