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ON THE OPERATION sup FOR SUBCATEGORIES OF MER

Á. CSÁSZÁR (Budapest)*, member of the Academy

Abstract. We study properties of the operation sup, defined for structures corresponding to different subcategories of MER, as merotopies, filter merotopies, contiguities, m-contiguities and closures. In particular, we examine commutativity of sup and the operation according to which a structure induces a structure of another type (as e.g. a merotopy induces a closure) and the inverse operations of the former.

0. Introduction

Let X denote an arbitrary (in general non-empty) set. We shall consider various topological structures on X that correspond to different subcategories of MER; instead of their original form, we use equivalent structures introduced essentially in [13]. So merotopies (see [15]) are replaced by *looseness*es [4], filter-merotopies [15] by *screen*s [5], contiguities [14] by ω-*looseness*es [4], instead of Čech proximities [1] we use the more general concept of m-*looseness*es ($2 \leqq m \in \mathbf{N}$) [4], also closures [1] are often described with the help of 2-*looseness*es.

In some sense, this paper is a continuation of [4] to [6]. Therefore, we assume that the definitions in [4] and [5] are familiar to the reader.

In order to simplify the formulation of some statements, we shall use the following conventions: let $\mathcal{S}_1$ denote the collection of all loosenesses on X, $\mathcal{S}_2$ the one of all screens on X, $\mathcal{S}_3$ that of all ω-loosenesses on X, $\mathcal{S}_4$ and $\mathcal{S}_5$ those of all n-loosenesses and m-loosenesses, respectively, on X ($2 \leqq m < n \in \mathbf{N}$), finally $\mathcal{S}_6$ that of all symmetric closures on X.

For each of the collections $\mathcal{S}_s$ ($1 \leqq s \leqq 6$) the operation sup is defined: if $\mathfrak{A}_i$ belongs to $\mathcal{S}_s$ for $i \in I \neq \emptyset$, $\sup\{\mathfrak{A}_i\}$ denotes the coarsest element of $\mathcal{S}_s$ finer than each $\mathfrak{A}_i$. For $s = 1, 3, 4, 5$, the construction of $\sup\{\mathfrak{A}_i\}$ is described in [4], 2.4. For $s = 2$, we have the following

LEMMA 0.1. *If each $\mathfrak{S}_i$ is a screen on X for $i \in I \neq \emptyset$, then* $\sup\{\mathfrak{S}_i\}$ *is given by* $\bigcap_{i \in I} \mathfrak{S}_i$.

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PROOF. Obviously $\mathfrak{S} = \bigcap_{i \in I} \mathfrak{S}_i$ is the largest screen contained in each of the screens $\mathfrak{S}_i$. $\square$

For $s = 6$, we can state:

LEMMA 0.2. *If c_i is a (symmetric) closure on X for $i \in I \neq \emptyset$, the coarsest (symmetric) closure c finer than each c_i is defined by*

$$x \notin c(A) \Leftrightarrow A \subset \bigcup_{1}^{n} A_k,\ x \notin c_{i_k}(A_k)\ (k = 1, \dots, n),\ i_k \in I. \tag{0.2.1}$$

PROOF. $c : \exp X \to \exp X$ defined by (0.2.1) is easily seen to be a (symmetric) closure on X. $x \notin c_i(A)$ for some $i \in I$ clearly implies $x \notin c(A)$. If c' is a closure on X and $x \notin c_i(A)$ implies $x \notin c'(A)$ then $x \notin c(A)$ implies $x \notin c'(A)$. $\square$

In what follows, we shall use the following conventions: i runs always over $I \neq \emptyset$. The term "closure" will be used in the sense of "symmetric closure". The character $\mathfrak{T}$ (possibly with subscript(s) or superscript(s)) will denote a looseness, $\mathfrak{S}$ a screen, $\mathfrak{R}$ an ω-looseness, $\mathfrak{N}$ an n-looseness, $\mathfrak{M}$ an m-looseness, always $2 \leqq m < n \in \mathbf{N}$, and c will always denote a (symmetric) closure, using always the underlying set $X \neq \emptyset$.

We shall also need the operations α_{st} from $\mathcal{S}_s$ to $\mathcal{S}_t$ defined in the following way. $\alpha_{12}(\mathfrak{T})$ is the screen $\mathfrak{S}(\mathfrak{T})$ induced by the looseness $\mathfrak{T}$ [7], $\alpha_{13}(\mathfrak{T})$ is the ω-looseness ${}^{\omega}\mathfrak{T}$ induced by $\mathfrak{T}$ [4], $\alpha_{14}(\mathfrak{T})$ or $\alpha_{15}(\mathfrak{T})$ is the n-looseness or m-looseness, denoted by ${}^{n}\mathfrak{T}$ or ${}^{m}\mathfrak{T}$, respectively, induced by $\mathfrak{T}$ [4], $\alpha_{16}(\mathfrak{T})$ is the closure $c(\mathfrak{T})$ induced by $\mathfrak{T}$ [4], $\alpha_{23}(\mathfrak{S})$ is the ω-looseness ${}^{\omega}\mathfrak{S}$ induced by the screen $\mathfrak{S}$ [5], $\alpha_{24}(\mathfrak{S})$ or $\alpha_{25}(\mathfrak{S})$ is the n-looseness or m-looseness, denoted by ${}^{n}\mathfrak{S}$ or ${}^{m}\mathfrak{S}$, respectively, induced by the screen $\mathfrak{S}$ [5], $\alpha_{26}(\mathfrak{S})$ is the closure $c(\mathfrak{S})$ induced by $\mathfrak{S}$ [5], $\alpha_{34}(\mathfrak{R})$ or $\alpha_{35}(\mathfrak{R})$ is the n-looseness ${}^{n}\mathfrak{R}$ or the m-looseness ${}^{m}\mathfrak{R}$ induced by the ω-looseness $\mathfrak{R}$ [4], $\alpha_{36}(\mathfrak{R})$ is the closure $c(\mathfrak{R})$ induced by $\mathfrak{R}$ [4], $\alpha_{45}(\mathfrak{N})$ is the m-looseness ${}^{m}\mathfrak{N}$ induced by the n-looseness $\mathfrak{N}$ [4], $\alpha_{46}(\mathfrak{N})$ is the closure $c(\mathfrak{N})$ induced by $\mathfrak{N}$ [4], $\alpha_{56}(\mathfrak{M})$ is the closure $c(\mathfrak{M})$ induced by the m-looseness $\mathfrak{M}$ [4].

Now we can simply describe the first purpose of this paper: we shall study the commutativity properties of the operations sup and α_{st}. Further questions arise concerning the commutativity of sup and the inverse operations of α_{st}; for the latters, we go into the details in Sections 2 and 3.

1. Commutativity of sup and α_{st}

Concerning our fundamental question, we can give sometimes positive, sometimes negative answer.