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ON SIMPLE ZEROS OF THE RIEMANN ZETA-FUNCTION IN SHORT INTERVALS ON THE CRITICAL LINE

J. STEUDING (Frankfurt)

Abstract. We calculate in a new way (following old ideas of Atkinson and new ideas of Jutila and Motohashi) the mean square of the product of a function $F(s)$, involving the Riemann zeta-function $\zeta(s)$, and a certain Dirichlet polynomial $A(s)$ of length $M = T^\theta$ in short intervals on $\sigma = a$ near the critical line: if $\theta < \frac{3}{8}$, then

$$\int_T^{T+H} \left| AF(a+it)\right|^2 dt = I(T,H) + O\Big(T^{\frac{1}{3}+\varepsilon} M^{\frac{4}{3}}\Big).$$

The main term $I(T,H)$ is well known, but the error term is much smaller than the one obtained by other approaches (e.g. $O\Big(T^{\frac{1}{2}+\varepsilon} M\Big)$). It follows from Levinson's method that the proportion of zeros of the zeta-function with imaginary parts in $[T, T+H]$ which are simple and on the critical line is positive, when $H \geqq T^{0.552}$.

1. Introduction

Let $s = \sigma + it$. Then the Riemann zeta-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\sigma > 1).$$

By the Euler product representation one has a first connection between the zeta-function and the prime numbers. Riemann discovered that $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ with residue 1, and satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s) \quad \text{with} \quad \chi(s) := \frac{(2\pi)^s}{2\Gamma(s)\cos\frac{\pi s}{2}}. \tag{1.1}$$

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$\chi(s)$ is a meromorphic function with only real zeros and poles; note that by Stirling's formula

(1.2)

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(i\left(\frac{\pi}{4} - t\log\frac{t}{2\pi e}\right)\right)\left(1 + O\left(\frac{1}{t}\right)\right) \quad (\sigma > 0,\ t \geqq 1).$$

It follows from the Euler product that there are no zeros in the halfplane $\sigma > 1$. The functional equation (1.1) implies the existence of simple zeros in $s = -2n$ ($n \in \mathbf{N}$), but no others in $\sigma < 0$. Nontrivial (non real) zeros $\varrho = \beta + i\gamma$ can only occur in the critical strip $0 \leqq \sigma \leqq 1$, but not on the real axis. Moreover we have $\zeta(\overline{s}) = \overline{\zeta(s)}$ by the reflection principle. Hence the nontrivial zeros lie symmetrically with respect to the real axis and the critical line $\sigma = \frac{1}{2}$. There are infinitely many nontrivial zeros: define $N(T)$ as the number of zeros $\varrho = \beta + i\gamma$ with $0 \leqq \beta \leqq 1$, $0 < \gamma \leqq T$. Riemann conjectured and von Mangoldt proved the Riemann–von Mangoldt-formula

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T). \tag{1.3}$$

Moreover, Riemann stated the famous and yet unproved hypothesis that all zeros in the critical strip have real part $\frac{1}{2}$ or equivalently the non-vanishing of $\zeta(s)$ in $\sigma > \frac{1}{2}$. The importance of the Riemann hypothesis lies in its connection with the distribution of primes: assuming this hypothesis one can show that the error term in the prime number theorem is as small as possible, so the Riemann hypothesis states, that the primes are distributed as uniformly as possible.

Van de Lune, te Riele and Winter localized the first 1 500 000 001 zeros without exception on the critical line; moreover they all turned out to be simple! By observations like this it is conjectured that all or at least almost all zeros of the zeta-function are simple. But, if $m(\varrho)$ denotes the multiplicity of the zero ϱ, it is only known that $m(\varrho) \ll \log|\gamma|$, which follows immediately from (1.3). Not only the vertical distribution of the zeros has arithmetical consequences, also their multiplicities as Cramér [8] observed.

What is known about the distribution of nontrivial zeros and their multiplicities? Using ideas of Hardy, Selberg was able to find a positive proportion of all zeros of $\zeta(s)$ on the critical line: if $N_0(T)$ denotes the number of zeros ϱ of $\zeta(s)$ on the critical line with $0 < \gamma \leqq T$, he proved

$$\liminf_{T\to\infty} \frac{N_0(T+H) - N_0(T)}{N(T+H) - N(T)} > 0$$

for $H \geqq T^{\frac{1}{2}+\varepsilon}$. Karatsuba improved this result to $H \geqq T^{\frac{27}{82}+\varepsilon}$ by technical refinement. However, the localized zeros are not necessarily simple.