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Fejér type theorems for Fourier–Stieltjes series

FERENC MÓRICZ

A b s t r a c t . A theorem of Fejér states that if a periodic function F is of bounded variation on the closed interval $[0, 2\pi]$, then the nth partial sum of its formally differentiated Fourier series divided by n converges to $\pi^{-1}[F(x+0) - F(x-0)]$ at each point x. The generalization of this theorem for Fourier–Stieltjes series of nonperiodic functions of bounded variation is also known. These theorems can be interpreted in such a way that the terms of the Fourier–Stieltjes (or Fourier) series of F determine the atoms of the finite Borel measure on the torus $\mathbf{T} := [0, 2\pi)$ induced by an appropriate extension of F (or by F itself in the periodic case). The aim of the present paper is to extend all of these results to the Cesàro as well as Abel–Poisson means of Fourier–Stieltjes (or Fourier) series of a nonperiodic (or periodic) function F of bounded variation. At the end, we sketch a possible extension of these results to linear means defined by more general kernels.

1. Introduction: Fejér's theorem revisited

In this paper, by F we always denote a function of bounded variation on the closed interval $[0, 2\pi]$. It is well known that such an F may have only simple discontinuities; that is, the left-hand limit $F(x-0)$ exists at each point $x \in (0, 2\pi]$, while the right-hand limit $F(x+0)$ exits at each point $x \in [0, 2\pi)$.

We recall that the Fourier–Stieltjes coefficients of F (also called the Fourier coefficients of dF) are defined by

$$(dF)^\wedge(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\, dF(t), \qquad k \in \mathbf{Z},$$

the integrals being Riemann–Stieltjes integrals. We write

$$dF(x) \sim \sum_{k\in\mathbf{Z}} (dF)^\wedge(k) e^{ikx} \tag{1.1}$$

and call this series the Fourier–Stieltjes series of F (also called the Fourier series of dF). Sometimes we denote this series by $S[dF]$. Let

$$s_n(dF, x) := \sum_{|k|\le n} (dF)^\wedge(k) e^{ikx}, \qquad n \in \mathbf{N},$$

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be the nth symmetric partial sum of series (1.1). The following theorem is attributed to Fejér (see, for example [6, (9.3) Theorem on p. 107]). However, in Fejér's paper [3] one can actually find its Corollary 1 presented at the end of this Section.

Theorem 1. *If F is a function of bounded variation on $[0, 2\pi]$, then for every $0 < x < 2\pi$ we have*

$$\lim_{n\to\infty} \frac{1}{n} s_n(dF, x) = \frac{1}{\pi}[F(x+0) - F(x-0)], \tag{1.2}$$

while for $x = 0$ or $x = 2\pi$ we have

$$\lim_{n\to\infty} \frac{1}{n} s_n(dF, x) = \frac{1}{\pi}[F(+0) - F(2\pi - 0) + c(F)], \tag{1.3}$$

where

$$c(F) := 2\pi(dF)^\wedge(0) = F(2\pi) - F(0). \tag{1.4}$$

Fejér certainly considered only the case when F is periodic. (See [3, p. 168], where he says the following: "Unter $D(0)$ verstehe ich $F(+0) - F(-0)$, d.h. $F(+0) - F(2\pi - 0)$.") Indeed, if F is a periodic function, then $c(F) = 0$ and the points $x = 0$ and $x = 2\pi$ do not play a distinguished role any longer. Consequently, (1.2) and (1.3) coincide in the periodic case, since then $F(-0) = F(2\pi - 0)$. However, the proof of (1.3) in the nonperiodic case requires a nontrivial modification of the proof of (1.2) (see [4]).

In the nonperiodic case, it is convenient to extend the definition of the given function F from $[0, 2\pi]$ to the whole real line $\mathbf{R}$ by requiring that

$$F(x + 2\pi) - F(x) = F(2\pi) - F(0) =: c(F), \qquad x \in \mathbf{R} \tag{1.5}$$

(see [6, p. 11]). Clearly, this extension does not change the values of $F(x)$ for $0 \le x \le 2\pi$, and this extension of F (which we denote also by F) is periodic if and only if $c(F) = 0$.

Now, with the extension of F given by (1.5) we associate the interval function $\mu(F, \cdot)$ defined by

$$\mu(F, I) := F(b - 0) - F(a + 0), \qquad \text{where } \; I := (a, b)$$

is any nonempty open interval on the torus $\mathbf{T} := [0, 2\pi)$. Clearly, the Borel measure induced by μ (which we denote also by μ) is finite if and only if F is of bounded variation on $[0, 2\pi]$.

It is not difficult to check that this induced measure μ is always periodic (in spite of the fact that the function F may not be periodic). Furthermore, if we denote by $\mu(F, x)$ the measure of the set consisting of the single point x, then we have

$$\mu(F, x) = F(x + 0) - F(x - 0). \tag{1.6}$$