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ASYMPTOTIC EXPANSION AND CONTINUED FRACTION FOR MATHIEU'S SERIES

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The series in question has the form

$$S(c)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+c^2)^2} \tag{1}$$

which is the object of the so-called Mathieu inequalities

$$\frac{1}{c^2+\frac{1}{2}}<S(c)<\frac{1}{c^2}$$

(for references see [6], pp. 360—362). It is easy to show that $S(c)$ has the Maclaurin expansion

$$S(c)=\sum_{i=0}^{\infty}(-1)^i(2i+2)\,\zeta(2i+3)\,c^{2i}\quad\text{for}\quad |c|<1, \tag{2}$$

where ζ (s) is the well-known zeta-function.

O. E. EMERSLEBEN [2] established a lower estimate and recently P. H. DIANANDA [1] has found an upper estimate for this series as follows

$$\frac{1}{c^2}-\frac{5}{16}\frac{1}{c^4}<S(c)<\frac{1}{c^2}-\frac{1}{(2c^2+2c+1)(8c^2+5c+3)}\,. \tag{3}$$

This suggests that Mathieu's series has an asymptotic expansion in the vicinity of $c=\infty$ of the form

$$S(c)\sim\frac{b_1}{c^2}+\frac{b_2}{c^4}+\frac{b_3}{c^6}+\dots,$$

where $b_1=1$ and $-5/16<b_2<-1/16$.

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Here we shall make use of an integral representation of $S(c)$ due to EMERSLEBEN [2]

$$S(c) = \frac{1}{c}\int_0^\infty \frac{x}{e^x - 1} \sin cx \, dx, \tag{4}$$

and the series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad \text{for} \quad |x| < 2\pi \tag{5}$$

(see [3], page 35) where B_n $(n = 0, 1, \ldots)$ are the Bernoulli numbers, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots, B_3 = B_5 = \ldots = 0$. In this note we shall prove the following result.

THEOREM. *The series* (1) *has the asymptotic expansion*

$$S(c) \sim \sum_{i=0}^{\infty} (-1)^i \frac{B_{2i}}{c^{2i+2}} = \frac{1}{c^2} - \frac{1}{6c^4} - \frac{1}{30c^6} - \ldots, \tag{6}$$

where B_{2i} $(i = 0, 1, \ldots)$ *is the* $2i^{\text{th}}$ *Bernoulli number.*

REMARK. Since

$$B_{2i} = (-1)^{i+1} (2\pi)^{-2i} 2(2i)! \, \zeta(2i) \sim (-1)^{i+1} (2\pi)^{-2i} 2(2i)!$$

for large i's (see [3], p. 53, formula 1.13 (22)), the sum in (6) is not convergent for any value of c. In spite of this fact the asymptotic expansion (6) is very useful to compute the values of $S(c)$ for large c's (see Fig. 2).

PROOF. Let the function $g(x)$ be introduced by $g(x) = x/(e^x - 1)$. Then integrating by parts in (4) we have

$$c^2 S(c) = 1 + \int_0^\infty g'(x) \cos cx \, dx.$$

By the Riemann—Lebesgue lemma the integral on the right-hand side tends to zero as c tends to infinity since $\int_0^\infty |g'(x)| \, dx < \infty$. Let us consider now the function

$$I_1(c) = c^4 \left[S(c) - \frac{1}{c^2} \right] = c^2 \int_0^\infty g'(x) \cos cx \, dx.$$

Integrating twice by parts we obtain

$$I_1(x) = [cg'(x) \sin cx]_0^\infty - \int_0^\infty g''(x) c \sin cx \, dx =$$

$$= [g''(x) \cos cx]_0^\infty - \int_0^\infty g'''(x) \cos cx \, dx = -B_2 - \int_0^\infty g'''(x) \cos cx \, dx.$$