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ON AN EXTENSION OF A LATTICE-CONTINUOUS MAPPING WITH EMBEDDABILITY APPLICATIONS

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Abstract

It is well known that if X and Y are completely regular T_2 spaces, then any continuous function, f, from X to Y, has a unique continuous extension, $\beta(f)$, from βX to βY, where βX and βY are the Stone—Čech compactifications of X and Y, respectively. This function plays an important role in Stone—Čech Theory, especially in questions pertaining to embeddability.

In this paper, we first extend this construction to general Wallman spaces, and then apply the results to extend well-known embeddability theorems.

Introduction

It is well known that if X and Y are completely regular T_2 spaces, then any continuous map $f{:}X \to Y$ has a unique continuous extension $\beta(f)\colon \beta X \to \beta Y$, where $\beta X, \beta Y$ are the Stone—Čech compactifications of X and Y, respectively. This map plays an important role in Stone—Čech Theory, especially in questions pertaining to embeddability. It is, therefore, important to extend this construction to general Wallman spaces and to consider analogous questions.

We do this first in a very general setting, without any appeal to representation theorems, and then apply the results to extend well-known embeddability theorems.

More specifically, let X, Y be abstract sets with lattices of subsets $\mathfrak{L}_1$, $\mathfrak{L}_2$, respectively, which are separating and disjunctive, and let $T\colon X \to Y$ be $\langle \mathfrak{L}_1, \mathfrak{L}_2\rangle$-continuous. If, in addition, T is surjective, and $T^{-1}(\mathfrak{L}_2)$ semiseparates $\mathfrak{L}_1$ (in particular, if T is $\langle \mathfrak{L}_1, \mathfrak{L}_2\rangle$-closed), then it is known [2] that $\hat{T}\colon IR(\mathfrak{L}_1) \to IR(\mathfrak{L}_2)$, where $\hat{T} = \mu T^{-1}$, extends T to the general Wallman spaces, and $\hat{T}$ is continuous with respect to the Wallman topologies. The problem arises,

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therefore, to generalize these results, for, in the important case where S is a subset of X and i: $S \to X$, where i is the injection and the lattices are $S \cap \mathfrak{L}$, and $\mathfrak{L}$, respectively, where $\mathfrak{L}$ is a given lattice of subsets of X, i is of course, not surjective, and the above results cannot be applied. We show, however, that with mild assumptions on the lattices, it is still possible to define an extension of T, to $\tilde{T}$: $IR(\mathfrak{L}_1) \to IR(\mathfrak{L}_2)$ which is continuous with respect to the Wallman topologies, and to utilize this mapping to extend known embeddability conditions.

The terminology and notation throughout are standard lattice theoretic ones; (e.g., see [1], [2] or [4]) we supply a few below that are less common, for convenience, and then proceed to the construction of $\tilde{T}$. Several applications are given to illustrate its usefullness. It should be clear that many more such applications are possible.

The set of all finitely additive measures on $\mathcal{A}(\mathfrak{L})$ is denoted by $M(\mathfrak{L})$. If the general element of $M(\mathfrak{L})$ is denoted by μ, then the set $\cap \{\mathfrak{L} \in L \mid \mu(\mathfrak{L}) = = \mu(X)\}$ is called the support of μ and is denoted by $S(\mu)$.

The subset of $M(\mathfrak{L})$ consisting of the zero-one valued measures is denoted by $I(\mathfrak{L})$. The subset of $I(\mathfrak{L})$ consisting of the $\mathfrak{L}$-regular measures is denoted by $IR(\mathfrak{L})$. If the general element of $\mathcal{A}(\mathfrak{L})$ is denoted by A, then the set $\{\mu \in I(\mathfrak{L}) \mid \mu(A) = 1\}$ is denoted by $V(A)$, and the set $\{\mu \in IR(\mathfrak{L}) \mid \mu(A) = 1\}$ is denoted by $W(A)$.

The set of all bounded $\mathfrak{L}$-continuous real valued functions on X is denoted by $C_b(\mathfrak{L})$. The set of all zero sets of X (determined by $\mathfrak{L}$), is denoted by $\mathfrak{Z}_X$. If the general element of $C_b(\mathfrak{L})$ is denoted by f, then the function h, determined by $h(v) = \int f dv$, $v \in IR(\mathfrak{L})$, is denoted by $\hat{f}$.

1. The mapping $\tilde{\mathbf{T}}$

PROPOSITION 1.1. *Consider any set X and any lattice $\mathfrak{L}$ of subsets of X such that $\mathfrak{L}$ is normal and disjunctive. The following statement is true. For every element L of $\mathfrak{L}$, $L = \cap \{\hat{L} \in \mathfrak{L} \mid$ there exists an element L_1 of $\mathfrak{L}$, such that $L \subset \subset L_1' \subset \hat{L}\}$.*

PROOF. Consider any element L of $\mathfrak{L}$. Denote the set whose intersection is equal to L, by $\mathfrak{S}$. Note $L \subset \cap \mathfrak{S}$. Next, show $\cap \mathfrak{S} \subset L$. Assume $\cap \mathfrak{S} \not\subset L$. Then, there exists an element x of X, such that $x \in \cap \mathfrak{S}$ and $x \notin L$. Consider any such x. Then, since $\mathfrak{L}$ is disjunctive, there exists an element $\tilde{L}$ of $\mathfrak{L}$, such that $x \in \tilde{L}$ and $L \cap \tilde{L} = \emptyset$. Consider any such $\tilde{L}$. Then, since $\mathfrak{L}$ is normal, there exist two elements L_1 and L_2 of $\mathfrak{L}$ such that $L \subset L_1'$, $\tilde{L} \subset L_2'$, and $L_1' \cap L_2' = = \emptyset$. Consider any such L_1, L_2. Then, $L \subset L_1' \subset L_2 \subset \tilde{L}'$. Hence, since $x \in \cap \mathfrak{S}$, $x \in L_2$. Consequently, $x \in \tilde{L}'$. (Contradiction.) Hence, $\cap \mathfrak{S} \subset L$. Consequently, $L = \cap \mathfrak{S}$.