Periodica Mathematica Hungarica Vol. 17 (1), (1986), pp. 65–69

AN ELEMENTARY MINIMAX INEQUALITY

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A minimax inequality between two real-valued functions f, g defined on the Cartesian product $X \times Y$ of nonempty sets X, Y is as follows:

$$(*) \qquad \inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y)$$

The $f = g$ case in $(*)$ is well-known as a minimax theorem for the function in question. In this connection the author proved recently an elementary but rather general result of minimax type [4]. Papers by S. Simons [5] and J. Kinder [2] called our attention to questions treated here: we shall prove a theorem which gives a sufficent condition on f and g to satisfy $(*)$. Our method is only a slight modification of argument used in [4]. It is somewhat surprising that the recent general situation can be treated by essentially known methods.

First we prove a minimax type result which is essentially the same as the Theorem in [4].

THEOREM 1. *Let f be a real-valued function on $X \times Y$ and let c_* be a real number or $-\infty$ such that the following properties are true:*

$$(1) \qquad \inf_{y \in Y} \max_{x \in A} f(x, y) \leq \max_{x \in A} \sum_{y \in B} \mu(y) f(x, y)$$

where $A \subset X$, $B \subset Y$ are finite sets and μ is a discrete probability measure on B.

$$(2) \qquad \inf_{y \in Y} \sum_{x \in A} \lambda(x) f(x, y) \leq c_*$$

where $A \subset X$ is a finite set and λ is a discrete probability measure on A.

AMS (MOS) subject classifications (1980). Primary 46A99; Secondary 26D99.
Key words and phrases. inf sup, sup inf, minimax inequality.

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Then for any finite set A in X we have

$$\inf_{y \in Y} \max_{x \in A} f(x, y) \leq c_* \tag{3}$$

Assume further the following property:
(4) *for any $c \in \mathbf{R}$, $c > c_*$ there exists a topology on Y and $x_0 \in X$ such that the sets*

$$K_x^c = \{y \in Y : f(x, y) \leq c\} \quad (x \in X)$$

are closed, $K_{x_0}^c$ is especially compact.
We have then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq c_* \tag{5}$$

Proof. Let $c \in \mathbf{R}$, $c > c_*$ and A, a finite set in X, be given. To prove (3) we need an y_0 in $\cap \{K_x^c : x \in A\}$ since then $f(x, y_0) \leq c$ holds for any x in A implying (3) by

$$\inf_{y \in Y} \max_{x \in X} f(x, y) \leq \max_{x \in A} f(x, y_0) \leq c.$$

We know from property (2) that $\inf_{y \in Y} f(x, y) \leq c_* < c$ holds for any $x \in X$, thus that K_x^c is never empty. Assuming now that $\cap \{K_x^c : x \in A\} = \emptyset$ we infere that for any $y \in Y$ there exists $x \in A$ with $f(x, y) > c$. Writing $A = \{x_1, \ldots, x_n\}$ and defining a function $\varphi: Y \to \mathbf{R}^n$ by

$$\varphi(y) := (f(x_1, y) - c, \ldots, f(x_n, y) - c) \quad (y \in Y),$$

we know that the range $\varphi(Y)$ of φ does not meet the negative cone $\mathbf{R}^n = \{(t_1, \ldots, t_n) : t_i \leq 0\ (1 \leq i \leq n)\}$ in $\mathbf{R}^n$. We state moreover that the convex hull $\mathrm{co}\varphi(Y)$ of the range of φ also does not meet the interior of $\mathbf{R}^n$. Otherwise there would be a finite set B in Y with a discrete probability measure μ on that such that

$$\sum_{y \in B} \mu(y)(f(x, y) - c) < 0 \quad (x \in A)$$

were satisfied. But then property (1) would imply

$$\inf_{y \in Y} \max_{x \in A} f(x, y) \leq \max_{x \in A} \sum_{y \in B} \mu(y) f(x, y) < c$$

showing an $y_0 \in Y$ with $f(x, y_0) < c$ for any $x \in A$, that is $y_0 \in \cap \{K_x^c : x \in A\}$.

Now using the separating hyperplane argument of [1], 2.5.1 we conclude a nonzero vector $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ in $\mathbf{R}^n$ such that

$$\sum_{i=1}^{n} \vartheta_i t_i \leq \sum_{i=1}^{n} \vartheta_i (f(x_i, y) - c) \quad (y \in Y;\ (t_1, \ldots, t_n) \in \mathbf{R}^n)$$