Periodica Mathematica Hungarica Vol. 18 (1), (1987), pp. 1—6

TWO NEGATIVE PARTITION RELATIONS

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Denote the ordinal ω_1 by ϱ. Hajnal has shown that the continuum hypothesis implies $\varrho \cdot \omega \nrightarrow (\varrho \cdot \omega, 3)^2$ and $\varrho \cdot \varrho \nrightarrow (\varrho \cdot \varrho, 3)^2$ [2]. In this paper, we consider set-theoretic hypotheses Γ and Δ (see Definition 2 below), whose conjunction is known to be weaker than the continuum hypothesis, and show that $\varrho \cdot \omega \nrightarrow (\varrho \cdot \omega, 3)^2$ follows from Γ and Δ (Theorem A), and that $\varrho \cdot \varrho \nrightarrow (\varrho \cdot \varrho, 3)^2$ follows from Γ (Theorem B).

We denote the cardinality of a set A by card (A), the set of all functions from A into B by $A \to B$, and the set of all finite subsets of A by $[A]^{<\omega}$. We call an ordered pair $\langle P, Q\rangle$ a *partition* of A if $A = P \cup Q$ and $P \cap Q = \emptyset$.

Suppose α and β are ordinals. Then $\alpha + \beta$ and $\alpha \cdot \beta$ are respectively the ordinal sum and product of α and β, and $[\alpha]^\beta$ denotes the set of all subsets of α of order type β.

DEFINITION 1. If α, β, γ are ordinals, then we use the notation $\alpha \to (\beta, \gamma)^2$ to abbreviate the statement that, for every partition $\langle P, Q\rangle$ of $[\alpha]^2$, either there is an $X \in [\alpha]^\beta$ such that $[X]^2 \subseteq P$, or there is an $X \in [\alpha]^\gamma$ such that $[X]^2 \subseteq Q$. We write $\alpha \nrightarrow (\beta, \gamma)^2$ for the negation of $\alpha \to (\beta, \gamma)^2$.

DEFINITION 2.

(a) Γ stands for the hypothesis that there exists a family $\mathscr{A} \subseteq [\varrho]^\omega$ such that card $(\mathscr{A}) = \varrho$ and $\forall X \in [\varrho]^\varrho\colon \exists A \in \mathscr{A}\colon A \subseteq X$.

(b) Δ stands for the hypothesis that there exists a family $\mathscr{G} \subseteq \omega \to \omega$ such that card $(\mathscr{G}) = \varrho$ and

$$\forall f \in \omega \to \omega\colon \exists g \in \mathscr{G}\colon \forall n < \omega\colon f(n) < g(n).$$

Both Γ and Δ are easy consequences of the continuum hypothesis. MA(ϱ) refutes both. Baumgartner has constructed a model of ZFC (Zermelo—

AMS (MOS) subject classifications (1980). Primary 03E05; Secondary 04A20.
Key words and phrases. Partition calculus, weakenings of the continuum hypothesis.

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Fraenkel set theory with the axiom of choice) in which both Γ and Δ hold but the continuum hypothesis fails ([1], Theorem 6.7, p. 428; Remark 6.12(b), p. 432).

Theorem A. *If the hypotheses Γ and Δ both hold, then* $\varrho \cdot \omega \nrightarrow (\varrho \cdot \omega, 3)^2$.

Proof. Assume Γ and Δ. By Γ there exist sets $A_\alpha \in [\varrho]^\omega$ $(\alpha < \varrho)$ such that

$$\forall X \in [\varrho]^\varrho \colon \exists \alpha < \varrho \colon A_\alpha \subseteq X. \tag{1}$$

Claim 1. *There exist functions $g_\zeta \in \omega \to [\varrho]^{<\omega}$ $(\zeta < \varrho)$ such that*

$$\forall f \in \omega \to \varrho \colon \exists \zeta < \varrho \colon \forall n < \omega \colon f(n) \in g_\zeta(n). \tag{2}$$

Proof. By Δ there exist families $\mathcal{G}_\alpha \subseteq \omega \to [\alpha]^{<\omega}$ $(\alpha < \varrho)$ such that, for every $\alpha < \varrho$, card $(\mathcal{G}_\alpha) = \varrho$ and

$$\forall f \in \omega \to \alpha \colon \exists g \in \mathcal{G}_\alpha \colon \forall n < \omega \colon f(n) \in g(n).$$

Let $\langle g_\zeta \mid \zeta < \varrho \rangle$ be an enumeration of the elements of $\bigcup \{\mathcal{G}_\alpha \mid \alpha < \varrho\}$. □

Claim 2. *There exist sets $Q(\xi, k, n) \in [\xi]^{<\omega}$ $(\xi < \varrho$ and $k < n < \omega)$ such that, for all ξ and k,*

$$\forall \zeta < \xi \colon \exists m \geq k \colon \forall n > m \colon \forall \alpha \in g_\zeta(n) \colon \\ [A_\alpha \subseteq \xi \Rightarrow Q(\xi, k, n) \cap A_\alpha \neq \emptyset], \tag{3}$$

and

$$\forall m > k \colon \forall \eta \in Q(\xi, k, m) \colon \forall n > m \colon \\ Q(\xi, k, n) \cap Q(\eta, m, n) = \emptyset, \tag{4}$$

where the functions g_ζ $(\zeta < \varrho)$ are as Claim 1.

Proof. Suppose $\xi < \varrho$ and $k < n < \omega$, and assume that the sets $Q(\xi', k', n')$ have been defined for all ξ', k', n' such that $\xi' < \xi$ or $[\xi' = \xi \ \& \ k' = k \ \& \ n' < n]$. Let $\langle h_i \mid i < \omega \rangle$ be an enumeration, independent of n, of all g_ζ such that $\zeta < \xi$. Note that $E = \bigcup \{h_i(n) \mid i < n\}$ is finite. Since

$$B = \bigcup \{Q(\eta, m, n) \mid k < m < n \ \& \ \eta \in Q(\xi, k, m)\} \in [\xi]^{<\omega},$$

one can choose $\gamma_\alpha \in A_\alpha - B$ for each $\alpha \in E$ with $A_\alpha \subseteq \xi$. Set

$$Q(\xi, k, n) = \{\gamma_\alpha \mid \alpha \in E \ \& \ A_\alpha \subseteq \xi\}.$$