Periodica Mathematica Hungarica Vol. 20 (1), (1989), pp. 85—87

A PROOF OF PARROTT'S THEOREM ON QUOTIENT NORMS

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Abstract

We prove, as an application of our positive extension argument, a theorem of Parrott concerning the quotient norm with respect to spaces of Hilbert space operators.

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A remarkable result due to Parrott ([1], Theorem 1) determines a quotient norm for bounded linear operators, sending a Hilbert space H into another one K with respect to the quotient map

$$p\colon B(H, K) \to B(H, K)/B(H_1, K_1)$$

where H_1 and K_1 stand for subspaces of H and K, respectively. Here an operator X from H to K corresponds to an operator from H_1 to K_1 if and only if X sends H_1 into K_1 and H_0, the orthocomplement of H_1 in H, into $\{0\}$. The orthocomplement of K_1 in K is K_0.

Parrott [1] proved for any $T \in B(H, K)$ that

$$||p(T)|| = \max \{||T \mid H_0||, ||T^* \mid K_0||\}$$

and moreover there is an operator $X \in B(H_1, K_1)$ such that

$$||p(T)|| = ||T + X||.$$

Here we want to give a proof of this theorem as a consequence of an extension process for positive operators (see [2], [3]).

The basic idea here is the following: the operator $S_0\colon H_0 \oplus K_0 \to H \oplus K$ given for $h \in H_0$, $k \in K_0$ by

$$S_0(h; k) = (T^*k, Th)$$

is symmetric and bounded by norm $m = \max(||T \mid H_0||, ||T^* \mid K_0||)$. S_0 has a self-adjoint extension S on $H \oplus K$ with the same norm m by Krein's Theorem

Mathematics subject classification number, 1980/85. Primary 46B05.
Key words and phrases. Quotient norm, Hilbert space operators, Banach space.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

(see [2]). To get S (see [3]), take the identity operator I_0 on $H_0 \oplus K_0$ and define a positive definite bilinear form on $H_0 \oplus K_0$ by

$$\langle (h;\, k),\, (h';\, k')\rangle = ((S_0 + mI_0)(h;\, k),\, (h';\, k')). \tag{1}$$

An easy calculation shows the following inequality

$$\|(T^*k + Mh;\, Th + Mk)\|^2 \leq 2m((S_0 + MI_0)(h;\, k),\, (h';\, k')). \tag{2}$$

Thus we have a Hilbert space L by factorization $H_0 \oplus K_0$ with respect to the kernel of $\langle\, ,\rangle$ and then by completion. Let $\hat{}$ denote the natural map from $H_0 \oplus K_0$ into (a dense subspace of) L. (2) yields a map V_0: $(H_0 \oplus K_0)\hat{} \to \to H \oplus K$:

$$V_0(\hat{h};\, \hat{k}) = (S_0 + mI_0)(h;\, k). \tag{3}$$

V_0 has a unique continuous linear extension V: $L \to H \oplus K$. Moreover, V has the crucial property

$$V^*(h';\, k') = (\hat{h}';\, \hat{k}') \quad (h' \in H_0,\ k' \in K_0) \tag{4}$$

as for any $h \in H_0$, $k \in K_0$ we have by (1) and (3)

$$\langle (\hat{h};\, \hat{k}),\, V^*(h';\, k')\rangle = (V(\hat{h};\, \hat{k}),\, (h';\, k')) = (V_0(\hat{h};\, \hat{k}),\, (h';\, k')) =$$
$$= ((S_0 + mI_0)(h;\, k),\, (h';\, k')) = \langle (\hat{h};\, \hat{k}),\, (\hat{h}';\, \hat{k}')\rangle,$$

i.e., VV^* extends $S_0 + mI_0$ to a positive operator on $H \oplus K$. $S = VV^* - mI$, I being the identity on $H \oplus K$, then extends S_0 to a self-adjoint operator S of the same norm m.

Stand J for the natural injection of H into $H \oplus K$ and let Q be the projection of $H \oplus K$ onto K. Then $X = QSJ - T$ satisfies all the asserted requirements. Indeed,

(i) $\|QSJ\| \leq \|Q\|\,\|S\|\,\|J\| = \|S\| = m$;
(ii) $Xh = 0$ for $h \in H_0$, QSJ extends $T\,|\,H_0$:

$$QSJh = QS(h;\, 0) = Q(0;\, Th) = Th$$

for $h \in H_0$;

(iii) X maps H_1 into K_1:

$$(Xh,\, k) = (QSJh,\, k) - (Th,\, k) = (h,\, J^*S^*Q^*k) - (h,\, T^*k) = 0$$

for any $h \in H_1$, $k \in K_0$ by $S^* = S$, and since

$$J^*SQ^*k = J^*S(0;\, k) = J^*S_0(0;\, k) = J^*(T^*k;\, 0) = T^*k.$$

This completes the proof.