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ON THE RING OF ENDOMORPHISMS OF A FINITELY GENERATED MULTIPLICATION MODULE

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§ 0. Introduction

Let R be a commutative ring with 1, and A is a (right) unitary R-module. The module A is said to be a multiplication module if every submodule N of A has the form AI for some ideal I of R [1].

Let End (A) be the ring of R-homomorphisms of A. In this note we show that if A is finitely generated (f.g.) multiplication module, then End (A) is a commutative ring, in fact, End (A) is isomorphic to R/ann (A) (See Th. 3.2).

Observe that End (A) is not commutative in general, for example, if A is free of rank $n > 1$, then End (A) is isomorphic to the ring of $n \times n$ matrices with entries in R, and this ring is not commutative.

It is known that a f.g. multiplication module with an annihilator generated by an idempotent is projective [9], [10]. The other main result of this paper shows that if the ring End (A) of a f.g. projective module A is commutative, then A is a multiplication module (See Th. 4.1).

In [2], Bergman called a module M hereditarily projective if M is projective and $l(M)$ is a projective ideal for each $l \in M^*$. In the last part of this paper we characterize f.g. hereditarily projective modules among f.g. projective modules using the commutativity of the ring End (M).

§ 1. Preliminaries

Let P be a f.g projective R-module generated by the components of the vector $\mathsf{U} = (a_1, a_2, \ldots, a_n)$. There exists an idempotent $n \times n$ matrix $M = [r_{ij}]$ with elements in R such that $\mathsf{U} = \mathsf{U} M$ and $\mathsf{U}^\perp = \operatorname{ann}(M)$, where

$$\mathsf{U}^\perp = \left\{ X = (x_1, x_2, \ldots, x_n) \in R^n \;\middle|\; \sum_{i=1}^{n} a_i x_i = 0 \right\}$$

and ann $(M) = \{X \in R^n \mid MX^t = 0\}$. (See [4], [5].)

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For each j, $1 \leq j \leq n$, define l^j: $P \to R$ as follows: If $a = \sum_{i=1}^{n} a_i x_i$, put

$$l^j(a) = \sum_{i=1}^{n} r_{ji} x_i.$$

In particular, $l^j(a_i) = r_{ji}$. It was proved in [9] that the set $\{l^j \mid 1 \leq j \leq n\}$ generates the module Hom $(P, R) = P^*$. Let A be a f.g. multiplication R-module generated by the components of $\mathsf{U} = (a_1, a_2, \ldots, a_n)$. There exists an $n \times n$ matrix $C = [c_{ij}]$ with elements in R such that

$$\mathsf{U} = \mathsf{U}C \tag{1}$$

$$\sum_{i=1}^{n} c_{ii} = 1 \tag{2}$$

$$a_k c_{ij} = a_j c_{ik} \quad \forall\, 1 \leq i, j, k \leq n. \tag{3}$$

(See [6], [7].)

Let $D =$ annann (A). For each j, $1 \leq j \leq n$, and each $d \in D$, define l^j_d: $A \to R$ as follows:

If $x = \sum_{i=1}^{n} a_i x_i \in A$, put

$$l^j_d(x) = d \sum_{i=1}^{n} c_{ji} x_i.$$

In particular, $l^j_d(a_i) = dc_{ji}$. It was proved in [8] that the set $\{l^j_d \mid d \in D, 1 \leq j \leq n\}$ generates the module Hom $(A, R) = A^*$.

In the next two sections we use similar ideas to define generators for Hom (A, A) and Hom (P, P).

§ 2. Generators for the module Hom (P, P)

Let P be a f.g. projective module. It is known that Hom (P, P) is a f.g. projective module (See [3]). In this section we construct a set of generators for Hom (P, P). Let $M = [r_{ij}]$ be a matrix associated with P, (See § 1). For each j, k, $1 \leq j, k \leq n$, define l^j_k: $P \to P$ as follows:

If $a = \sum_{i=1}^{n} a_i x_i$, put

$$l^j_k(a) = \sum_{i=1}^{n} a_k r_{ji} x_i = a_k \sum_{i=1}^{n} r_{ji} x_i.$$

In particular, $l^j_k(a_i) = a_k r_{ji}$. It follows immediately from the properties of the matrix M that l^j_k is well defined R-homomorphism.