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SOME METRICAL OBSERVATIONS ON THE APPROXIMATION OF AN IRRATIONAL NUMBER BY ITS NEAREST MEDIANTS

H. JAGER (Hoofddorp)

1. Approximation by nearest mediants

(1.1) NOTATIONS. Let $\omega \in \Omega := [0,1]\backslash\mathbf{Q}$. Denote the regular continued fraction expansion of ω by

$$\omega = [0; a_1(\omega), a_2(\omega), \ldots, a_n(\omega), \ldots]$$

and the corresponding sequence of convergents by

$$\left(\frac{p_n(\omega)}{q_n(\omega)}\right)_{n\geq -1}.$$

The sequence of *approximation coefficients* $(\theta_n(\omega))_{n\geq -1}$ of ω is defined by

$$\theta_n(\omega) := q_n(\omega)|\omega q_n(\omega) - p_n(\omega)|, n \geq -1. \tag{1.2}$$

In the sequel we will often omit the ω in these notations and write simply

$$\omega = [0; a_1, a_2, \ldots, a_n, \ldots], \frac{p_n}{q_n}, \theta_n.$$

(1.3) DEFINITION. The sequence of *mediants* of an irrational number ω is the sequence of irreducible fractions of the form

$$\frac{bp_n(\omega) + p_{n-1}(\omega)}{bq_n(\omega) + q_{n-1}(\omega)}, n \geq 0,\ b = 1, 2, \ldots, a_{n+1}(\omega) - 1,$$

ordered in such that a way that the denominators form an ascending sequence, compare [9], p.26.

In this paper we will especially be concerned with the so-called *extreme* or *nearest mediants*, i.e. those mediants formed with $b = 1$ and $b = a_{n+1} - 1$.

(1.4) DEFINITION. For $\omega \in \Omega$ the sequences

$$\left(\frac{A_n(\omega)}{B_n(\omega)}\right)_{n\geq 1} \text{ and } \left(\frac{C_n(\omega)}{D_n(\omega)}\right)_{n\geq 1}$$

are defined by

$$\frac{A_n(\omega)}{B_n(\omega)} := \frac{p_n(\omega) + p_{n-1}(\omega)}{q_n(\omega) + q_{n-1}(\omega)}, n \geq 1$$

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and

$$\frac{C_n(\omega)}{D_n(\omega)} := \frac{(a_n(\omega)-1)p_{n-1}(\omega)+p_{n-2}(\omega)}{(a_n(\omega)-1)q_{n-1}(\omega)+q_{n-2}(\omega)} = \frac{p_n(\omega)-p_{n-1}(\omega)}{q_n(\omega)-q_{n-1}(\omega)},\ n \geq 1.$$

Note that the elements of the sequence $(A_n/B_n)_{n\geq 1}$ are mediants or convergents p_k/q_k with k such, that $a_k = 1$. For the other sequence we have a similar property.

The corresponding sequences $(\varrho_n(\omega))_{n\geq 1}$ and $(\sigma_n(\omega))_{n\geq 1}$ of *approximation coefficients* are defined by

$$\varrho_n(\omega) := B_n(\omega)|\omega B_n(\omega) - A_n(\omega)|,\ n \geq 1$$

and

$$\sigma_n(\omega) := D_n(\omega)|\omega D_n(\omega) - C_n(\omega)|,\ n \geq 1,$$

compare (1.2). Again we will frequently omit the ω in these notations.

(1.5) REMARK. The two sequences

$$\left(\frac{A_n}{B_n}\right)_{n\geq 1} \text{ and } \left(\frac{C_n}{D_n}\right)_{n\geq 1}$$

are both converging to ω. They show a number of similarities in their convergence to ω with the sequence of convergents

$$\left(\frac{p_n}{q_n}\right)_{n\geq 1}$$

to which we will draw the attention in the first part of this paper. To begin with there is the alternating way in which they converge to ω; it is easy to verify that

$$\frac{A_1}{B_1} < \frac{A_3}{B_3} < \cdots < \omega < \cdots < \frac{A_4}{B_4} < \frac{A_2}{B_2}$$

and

$$\frac{C_2}{D_2} < \frac{C_4}{D_4} < \cdots < \omega < \cdots < \frac{C_3}{D_3} < \frac{C_1}{D_1},$$

see also lemma (2.12).

For the sequence $(p_n/q_n)_{n\geq 1}$ we have the two classical theorems of P. Lévy, see [1]:

(1.6) $$\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12\log 2}, \quad \text{for almost all } \omega$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \left|\omega - \frac{p_n}{q_n}\right| = -\frac{\pi^2}{6\log 2}, \quad \text{for almost all } \omega.$$

Similar results are easy to prove for the sequences $(A_n/B_n)_{n\geq 1}$ and $(C_n/D_n)_{n\geq 1}$. One finds that

$$\lim_{n\to\infty} \frac{1}{n} \log B_n = \lim_{n\to\infty} \frac{1}{n} \log D_n = \frac{\pi^2}{12\log 2}, \quad \text{for almost all } \omega$$