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GENERALIZED RAMSEY THEORY FOR GRAPHS, I. DIAGONAL NUMBERS

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Dedicated to the memory of ALFRÉD RÉNYI

0. Approach

We use the notation and terminology of [12]. The ramsey number $r(m, n)$ as traditionally studied in graph theory may be defined as the minimum number p such that every graph with p points which does not contain the complete graph K_m must have n independent points. Alternatively, it is the smallest p for which every coloring of the lines of K_p with two colors, green and red, contains either a green K_m or a red K_n. Thus the *diagonal ramsey numbers* $r(n, n)$ can be described in terms of 2-coloring the lines of K_p and regarding K_n as a forbidden monochromatic subgraph without regard to color.

This viewpoint suggests the more general situation in which an arbitrary graph G has a c-coloring of its lines and the number of monochromatic occurrences of a forbidden subgraph F (or of a forbidden family of graphs) is calculated. A host of problem areas within graph theory can be subsumed under such a formulation. These include the line-chromatic number, in which the 3-point path is forbidden. The arboricity of G involves forbidding all cycles. The thickness of a graph forbids the Kuratowski graphs. Complete bipartite graphs can be taken for both G and F, and so can cubes Q_n and Q_m.

There has long been a sentiment in graph theory that there is an intimate relationship between extremal graph theory and ramsey numbers. It does not appear possible to derive either TURÁN's Theorem or RAMSEY's Theorem from the other. However, extremal bipartite graph theory does in fact imply the bipartite form of RAMSEY's Theorem. The mystery behind these implications is revealed by Theorem 1.

A combinatorial technique used by ERDŐS to find a lower bound for diagonal ramsey numbers $r(n, n)$ is extended to generalized ramsey numbers for arbitrary graphs and forbidden subgraphs.

1. Introduction

Let $\mathcal{F}$ be a family of graphs, G a given graph, and c a positive integer. We denote by $R(G, \mathcal{F}, c)$ the greatest integer n with the property that, in every coloration of the lines of G with c colors, there are at least n monochromatic occurences of a member of $\mathcal{F}$. Without any loss of generality, we can assume that every forbidden subgraph $F \in \mathcal{F}$ has no isolated points. Among the typical families $\mathcal{F}$ of forbidden subgraphs, we mention the family $\mathcal{C}$ of all cycles, the family $\mathcal{C}_0$ of odd cycles, and the family $\mathcal{K}$ of Kuratowski graphs, namely those homeomorphic to K_5 or $K_{3,3}$. If $\mathcal{F}$ contains just one forbidden subgraph F then we write simply $R(G, F, c)$ instead of $R(G, \{F\}, c)$.

The numbers $R(G, \mathcal{F}, c)$ are useful in formulating various graph theoretical problems and results. For instance, the four color conjecture states that

$$R(G, \mathcal{K}, 1) = 0 \text{ implies } R(G, \mathcal{C}_0, 2) = 0,$$

i.e., that every planar graph is the line-disjoint union of two bigraphs. An equivalent formulation of the four color conjecture is:

For every bridgeless cubic planar graph G, $R(G, P_3, 3) = 0$, in other words, every bridgeless cubic planar graph is 1-factorable. Vizing [18] proved that

$$R(G, P_3, \Delta + 1) = 0$$

when he showed that the line-chromatic number of every G is either Δ or $\Delta + 1$. Obviously, the thickness of a graph G is the minimum n such that

$$R(G, \mathcal{K}, n) = 0.$$

Similarly, the arboricity of G is the minimum n such that

$$R(G, \mathcal{C}, n) = 0.$$

Nash-Williams [14] proved that the arboricity of a graph G is equal to

$$\max_{n \leq p} \left\{ \frac{q_n}{n-1} \right\}$$

where q_n is the maximum number of lines spanned by n points. When Ramsey's theorem [15] is specialized to graphs, it asserts that given any positive integers m and c there is always an integer $n = n(m, c)$ such that

$$R(K_n, K_m, c) > 0.$$

Similarly, Erdős and Rado [6] proved for complete bigraphs that given any positive integers m and c there is always an integer $n = n(m,c)$ such that

$$R(K_{n,n}, K_{m,m}, c) > 0;$$

this is sometimes called the theorem on polarized partition relations.

We now present a brief summary of results involving R-numbers. Erdős and Szekeres [8] proved that

$$n \geq \binom{2m-2}{m-1} \quad \text{implies} \quad R(K_n, K_m, 2) > 0,$$